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ABSTRACT

The Sequential Mathematics Sequence provides students with an integrated course of study, including lessons which interweave algebra, geometry, logic, probability, and mathematical systems. This handbook is designed to meet the needs of teachers who will be teaching Course III in the sequence. The handbook, for use with staff development programs, has seven main sections: (1) program scope and sequence; (2) comparison of sequential mathematics Course III and 11th year mathematics; (3) teaching functions; (4) teaching geometric transformations; (5) teaching probability and statistics; (6) teaching logarithmic, exponential and trigonometric functions; and (7) standards from rating regents examinations. (PK)

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ED289694

Sequential Mathematics Course III

Volume One

Staff Development Program 1986-1987

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FOREWORD

The Division of Curriculum and Instruction welcomes you to our 1986-87 Staff Development Program. As there are two possible formats for this year's program, the Program, Scope and Sequence have been designed so that they can be easily adapted to accommodate either of the 12-hour programs. This handbook for Sequential Mathematics Course III meets the needs of teachers who will be teaching the course this fall. Each unit is planned for a four-hour session. The same unit can be used for two two-hour workshops. It is our hope that the ideas generated in these sessions will be used in the classroom to stimulate learning. Sequential Mathematics Course III is the third year of the New York State Regents Diploma requirement for the Regents Action Plan.

There will be follow-up workshops next semester to continue the process of implementing Sequential Mathematics Course III stressing additional areas of instruction and subject-matter. Your evaluation and suggestions will be included in the next group of workshops. Some of the areas to be developed in the follow-up program will include questioning techniques, student activities, improving student study skills and math games to promote retention of the mathematical concepts and procedures developed in the Sequential Mathematics Course III classes. It is hoped this course of study will lead to more students electing fourth and fifth year mathematics courses.

We wish to thank all those who have participated in developing materials for the various workshops. We hope that through our combined efforts in sharing new ideas and skills, mathematics education in New York City will continue to provide challenging and supportive instruction for all our students.

Charlotte Frank

Charlotte Frank
Executive Director
Division of Curriculum and Instruction

ACKNOWLEDGEMENTS

The 1986-87 Staff Development Program for Sequential Mathematics Course III was developed as a project of the Mathematics Unit of the Division of Curriculum and Instruction, Charlotte Frank, Executive Director.

Morris Freedman, Director of the Office of Curriculum Development and Support, provided overall supervision of the project.

The project was initiated under the supervision of Lorraine Largmann, Director of Mathematics. Maria DeSalvio, Assistant Director of Mathematics, gave impetus and guidance to this project by supervising the following planning, writing and editing teams.

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SEQUENTIAL MATHEMATICS III

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PROGRAM

SCOPE AND SEQUENCE

STAFF DEVELOPMENT PROGRAM FOR SEQUENTIAL MATHEMATICS, COURSE III
SCOPE AND SEQUENCE

INFORMATIONAL UPDATE	COMPARISON OF SEQUENTIAL MATHEMATICS COURSE III AND ELEVENTH YEAR MATHEMATICS
	Performance Objectives: Participants will be able to...
Overview of the Content of Sequential Mathematics, Course III	State the similarities and differences between topics (scope and sequence) of Sequential Mathematics III and Eleventh Year Mathematics curricula.
Functions	State the various definitions of functions and give examples of each type. Find inverses of functions, and find the domains and ranges of given functions. Find the composition of two given functions Describe the use of functions as a unifying concept in teaching Sequential Mathematics, Course III. Plan and adapt specific lessons on functions, including linear functions, trigonometric functions, exponential functions and absolute value functions.
Geometric Transformations	Define and state the various geometric transformations. Solve motivational problems involving transformations in the plane. Plan and adapt lessons on transformations in the curriculum guide.

TEACHING PROBABILITY, THE EXPONENTIAL FUNCTIONS AND TRIGONOMETRY IN SEQUENTIAL MATHEMATICS, COURSE III

	Performance Objectives: Participants will be able to...
The Exponential and Logarithmic Functions	Prepare visual materials for teaching exponential functions, trigonometric functions and probability.
Probability, Combinations and Permutations	Solve simple probability problems from Sequential Mathematics, Course III. Use game-playing activities as a problem solving strategy. Prepare suitable alternative motivations and "do now" activities.
Trigonometric Functions and Inverse Trigonometric Functions	Plan and adapt lessons on exponential functions and trignonmetric functions, their inverses and probability.
Testing and Resources	Analyze sample tests on geometric transformations. List references and identify available resources for teaching Sequential Mathematics Course III.

TEACHING STATISTICS AND MARKING SAMPLE REGENTS EXAMINATIONS QUESTIONS IN SEQUENTIAL MATHEMATICS COURSE III

	Performance Objectives: Participants will be able to...
The Summation Symbol	Define and use the sigma notation.
	Solve typical problems using the summation symbol in statistics and algebra.
Measures of Central Tendency and Measures of Dispersion	Define and use mean, median and mode and various measures of dispersion.
Standard Deviation and Normal Distribution	Illustrate the normal distribution curve with the mean and three standard deviations to the left and right of the mean.
	Analyze and use percentiles on the normal distribution curve.
	Develop suitable motivations for the lessons on statistics.
	Plan and adapt specific lessons on the statistics unit in Sequential Mathematics Course III.
Problem Solving	Analyze and solve non-routine problems.
	Anticipate student errors in solving specific problems.
Standards for Rating Regents Examination Papers	Grade sample Regents Examination questions (Part II) using the guidelines suggested by the New York State Education Department.

THREE-YEAR SEQUENCE FOR HIGH SCHOOL

mathematics

COURSE III

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Bureau of General Education Curriculum Development
Albany, New York

Reprinted 1982

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FOREWORD

This is the third in a series of three publications designed for the new *Three-Year Sequence for High School Mathematics* program consisting of three unified courses - *Course I, Course II,* and *Course III.* Currently, this sequence provides an alternate to the regular Regents sequence of ninth, tenth, and eleventh year mathematics. Examinations will be available each year for each course, and students who successfully complete the courses will be credited with a Regents sequence in mathematics.

In June 1972, an Advisory Committee was convened by Frank Hawthorne, then Chief of the Bureau of Mathematics Education, to consider the development of a three-year high school mathematics program which would bring together the various branches of mathematics that historically have been treated independently. Members of this committee included: Robert Atkins, John Adams High School, New York City; Donald Holquist, then of Mineola High School, now at Baldwin High School; Anthony Prindle, Linton High School, Schenectady; Ralph Roberts, Pelham Memorial High School; Harry Ruderman, Hunter College High School; George Salterelli, Amherst High School; Stephen Willoughby, New York University; and Sister Ann Xavier, Nazareth High School, Rochester.

Using the philosophical deliberations and tentative outline developed by the committee, *Course III* was organized and written by Sister Ann Xavier of the committee and Richard Klutch of Hunter College High School, working under the direction of Fredric Paul, Chief of the Bureau of Mathematics Education. John Matmillo of the Wappingers Central Schools reviewed the material. In September 1976, 40 school districts began offering the course.

As a result of a series of meetings with classroom teachers during 2 years of teaching the course, together with their written evaluations at the end of that time, *Course III* was revised during the summer of 1978. Janet Burt of Clinton High School and Sherman Taishoff of Harrison High School analyzed the evaluations and made recommendations which were incorporated into the revision of the materials by the original writers.

The final draft was prepared and edited by Mr. Paul and Aaron Buchman, Associate in Mathematics Education. Robert F. Zimmerman, Associate in Secondary Curriculum, represented the Bureau of General Education Curriculum Development on the committee and served as the liaison with the Bureau of Mathematics Education. The final manuscript was prepared in the Bureau of General Education Curriculum Development.

The curriculum defined in this publication supports the State Plan for Basic Skills and was reprinted, in part, with funds from ESEA Title II Basic Skills.

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COURSE III

Scope and Content

I. Complex Numbers (recommended time: 25 days)

- A. Further study of the real number system
- B. Extension of algebraic techniques
- C. Introduction of the complex number system
- D. Quadratics with complex roots
- E. Absolute value: equations and inequalities

II. Relations and Functions (recommended time: 25 days)

- A. Definitions and properties
- B. Exponential and logarithmic functions
- C. Common logarithms and computation

III. Circular Functions (recommended time: 60 days)

- A. Further work with the geometry of the circle
- B. Circular functions and their graphs
- C. Trigonometric identities and equations
- D. Applications to area, Law of Sines, Law of Cosines, right triangle trigonometry

IV. Transformation Geometry (recommended time: 20 days)

- A. Review of mapping
  1. Definitions, one-to-one, identity mappings, inverse mappings
  2. Some geometric examples
- B. Line Reflections
  1. Considered as mappings
  2. Properties "preserved" by line reflections
  3. Line symmetry
  4. Coordinates of image of point (x, y) under reflection in special lines of coordinatized plane

C. Translations

1. Considered as mappings
2. Properties "preserved" by translations
3. Translations as composition of line reflections
4. Coordinates of image of point (x, y) under translation

D. Rotations

1. Considered as mappings
2. Properties "preserved" by rotations
3. Rotations as composition of line reflections
4. Coordinates of image of point (x, y) under special rotations
5. Rotational symmetry

E. Isometries

1. Definition
2. Relation to congruence
3. Direct isometries
4. The three-line reflection theorem
5. Dilations as examples of transformations which are not isometries
6. Coordinates of image of point (x, y) under dilation
7. Relation to symmetry

F. Applications to Group Theory

1. The identity transformation
2. Transformations whose composition is the identity
3. The group of translations
4. The group of rotations
5. The groups of isometries; of direct isometries

V. Probability and Statistics (recommended time: 20 days)

A. Review of previous work with probability, combinatorics

B. Probability of exactly r successes in n trials

C. The Binomial Theorem

D. Review of previous work with statistics

E. Measures of dispersion:

1. range, mean absolute deviation
2. variance
3. standard deviation

F. Σ - notation

VI. Optional Topics

A. Polar coordinates

B. Number theory

C. Mathematical induction

D. Sequences and series

E. Matrices

F. Synthetic substitution, remainder theorem, and factor theorem

I. Complex Numbers

A. In *Courses I* and *II*, students investigated mathematical systems involving sets of elements, operations, and properties of these operations. In *Course III*, teachers may begin a more structure-oriented discussion of the real number system by considering the set of whole numbers (the set of natural numbers and zero). These may be regarded as undefined elements or may be defined in terms of some more basic concept, such as sets. Together with the operations of addition and multiplication, the whole numbers form a system in which the closure, commutative, associative, distributive, and cancellation properties are valid. Zero is the additive and one the multiplicative identity.

In this number system, subtraction is defined so that $a - b = x$ if, and only if, $b + x = a$. But in the system of whole numbers, the equation $b + x = a$ does not always have a solution, and, therefore, subtraction is not always possible. This motivates the introduction of the set of integers.

For all a in the set of whole numbers, $-a$, the additive inverse of a, is the solution of the equation $a + x = 0$. The union of the set of whole numbers and the set of their additive inverses is called the set of integers.

In order that the integers, as a number system, retain the basic properties of the whole number system, it is necessary to define addition and multiplication is some way similar to the following: for all a, b, whole numbers,

$$(-a) + (-b) = -(a + b)$$

$$(-a) + (b) = \begin{cases} b - a \text{ if } b - a \text{ is a natural number} \\ -(a - b) \text{ if } a - b \text{ is a natural number} \end{cases}$$

$$(-a)(-b) = ab$$

$$(-a)(b) = -(ab)$$

The equation, $b + x = a$, always has a solution and that solution can be shown to be $a - b = a + (-b)$.

Division is defined in a manner similar to subtraction. If a and b are integers and $b \neq 0$, the quotient $\frac{a}{b}$ is equal to x if and only if $bx = a$. But this equation does not always have a solution in the set of integers. Again, it is necessary to introduce a new set of numbers, the set of all ratios of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Equality of elements in this set and operations of addition and multiplication are defined so that the properties of the whole numbers and of the integers hold:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

This new set forms a number system in itself, some of whose elements are integers. To demonstrate that this is the case, consider the equation $1x = a$.

$x = a$, since 1 is the identity element for multiplication

$\therefore x = \frac{a}{1}$ by the definition of division

The integer a is equal to the ratio $\frac{a}{1}$

Also, consider the equation $cx = ca$, $c \neq 0$

$x = a$ by the cancellation law

$x = \frac{ca}{c}$ by the definition of division

$\therefore a = \frac{ca}{c}$

Using this new set of numbers, called the rational numbers, the equation $bx = a$ always has a solution if $b \neq 0$. In particular, the equation $ax = 1$ has a solution for all $a \neq 0$ and that solution, $\frac{1}{a}$, is called the multiplicative inverse (or reciprocal) of a. It is now possible to verify that the set of rational numbers under the operations of addition and multiplication form a field. (See *Course II*, page 18.)

While the transition from natural numbers and whole numbers to integers and from integers to rational numbers is relatively simple, the introduction of the real number system is not. If the approach used before is employed, the equation $x^2 = a$ could be considered. This equation does not always have a solution in the set of rational numbers. The introduction of a set of numbers which will solve this equation for $a > 0$ is very different from the set of numbers which will solve the equation for $a < 0$ and in neither case do we have a basis for the set of real numbers. So while an equation such as $x^2 = 2$ indicates the need for a set of numbers which includes numbers which are not rational, it does not help us to define the reals. At this level of mathematical maturity, most students do not have the

mathematical background for a complete definition. The most common approach is an appeal to the number line. The real numbers are described in terms of a 1 to 1 correspondence with the points on a line. Such a description could motivate a geometric interpretation of addition, subtraction, multiplication, and division. In addition, this approach might suggest an informal discussion of "greater than" in terms of "to the right of, on the number line."

A more formal approach to the ordering of the real numbers uses positive as an undefined term and the statement that the product of two positive numbers is a positive number as a postulate. Then, for all real numbers a and b, $a > b$ if and only if there exists a positive real number x such that $a = b + x$. Using this definition, the following statements can be proved.

If a is positive, then $a > 0$ since $a = 0 + a$

If $a > 0$, then $0 > -a$ since $0 = -a + a$

If $a > b$, then $a + c > b + c$ since $a = b + x \longrightarrow a + c = b + c + x$

If $a > b$ and $c > 0$, then $ac > bc$ since $a = b + x \longrightarrow ac = bc + xc$

If $0 > a$ and $c > 0$, then $0 > ac$ since $0 > a \longrightarrow 0 = a + x$

and $c > 0 \longrightarrow c = 0 + c$

$\therefore 0 = ac + xc$

If $a > b$ and $0 > c$ then $bc > ac$ since $a = b + x$

$ac = bc + xc$

$0 > xc$

$-xc > 0$

$ac + (-xc) = bc$

$bc > ac$

The teacher may find it convenient to introduce into this discussion the definition of negative (a is negative if and only if $0 > a$) and the definition of less than ($a < b$ if and only if $b > a$).

Another common characterization of the reals is in terms of "infinite" decimals. The student has had no previous work with sequences or series, and many important concepts involved in working with numbers in this form would have to be accepted without proof if the teacher chooses to present the real numbers

in this way. However, such a presentation can provide a relatively accessible distinction between rational and irrational numbers and highlight the existence of irrational numbers which are not radicals or in any way dependent on the solution of an algebraic equation.

B. As the discussion of the real number system develops, the teacher will find ample opportunity to use the properties under discussion to strengthen students' ability to rewrite algebraic expressions in simpler form. The distributive property of multiplication over addition should be familiar to the student and its use in factoring recognized. Students have had limited experience with algebraic fractions. Practice should lead to competence in writing equivalent fractions, multiplying and dividing algebraic expressions in fraction form, and in the simplification of complex fractions. Students should be able to simplify fractions involving radicals and, in particular, should be able to rationalize the denominator of a fraction.

C. The complex number system can be introduced in a manner similar to that used for the integers and the rational numbers. The fact that $x^2 = -1$ has no solution in the set of real numbers, motivates the definition $i = \sqrt{-1}$. The set of complex numbers is defined as the set of all numbers of the form $a + bi$, where a and b are real numbers. The operations of addition and multiplication are defined so that the basic properties hold. Students should develop facility in working with these operations while verifying that the complex numbers, together with addition and multiplication as defined, form a field.

It should be brought to the student's attention that unlike the fields of rational numbers and real numbers the field of complex numbers cannot be ordered. Those complex numbers which are not real numbers cannot be classified as positive or negative even when the real numbers involved in the formation of the number can be so classified. For example, if 3i were positive or negative, its square would be positive, which, of course, it is not. Also, it is false that 5i is greater than 2i, since $5i - 2i = 3i$ is not positive.

D. The solution of quadratic equations by the quadratic formula was studied in *Course II*. At that time only equations with real solutions were considered. These equations should be reviewed and equations with complex roots should be considered. This study should include the use of the discriminant to determine the nature of the roots, the relationship between the quadratic equation and the second degree polynomial function and the relationship between the nature of the roots and the graph of the function.

It can be shown, either by examples of factored quadratics and/or computation with

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

that for any quadratic $ax^2 + bx + c = 0$, the sum of the roots equals $\frac{-b}{a}$ and the products of the roots equals $\frac{c}{a}$. Example: Write a quadratic equation having the roots $(3 - 2i)$ and $(3 + 2i)$.

$$x_1 + x_2 = \frac{-b}{a} \qquad x_1 \cdot x_2 = \frac{c}{a}$$

$$(3 - 2i) + (3 + 2i) \qquad (3 - 2i)(3 + 2i)$$

$$6 = \frac{-b}{a} \qquad 9 - 4i^2$$

Let $a = 1$ $\qquad 9 + 4$

then $b = -6$ $\qquad 13 = \frac{c}{a}$

since $a = 1$

$c = 13$

$\therefore$ An equation would be $x^2 - 6x + 13 = 0$

E. For students who have not already studied absolute values, it should be introduced and formally defined at this time.

Here is one definition:

$$\begin{cases} |x| = x & \text{if } x \geq 0 \\ |x| = -x & \text{if } x < 0 \end{cases}$$

This definition can be used to solve first degree absolute value equations.

Example: Solve: $|2x - 3| = 5$

If $2x - 3 \geq 0$ $\qquad$ If $2x - 3 < 0$

$2x - 3 = 5$ $\qquad -(2x - 3) = 5$

$2x = 8$ $\qquad -2x + 3 = 5$

$x = 4$ $\qquad -2x = 2$

$x = -1$

The solution set for the equation is $\{ -1, 4 \}$

Simple inequalities using absolute value can also be pursued at this time. Many teachers will wish to limit their discussions to problems like $| x | < 4$ or $| x | > 4$, the solution to which, with appropriate illustrations on the number line, can be related to problems involving quadratics (such as $x^2 < 16$). In other classes, teachers may extend this work to include more complex inequalities such as $| 2x - 3 | < 5$ or quadratic inequalities of the type $x^2 - 4x - 5 < 0$.

II. Relations and Functions

A. Throughout the study of mathematics, as well as in non-mathematical situations, students have encountered the concept of relations. In the section dealing with Euclidean geometry in *Course II*, this concept was introduced in order to discuss the idea of an equivalence relation. Since the concept of relation underlies and simplifies the expression of many mathematical ideas, it should now be considered in greater detail.

Each time use is made of a mathematical expression like $2x + 1$, a relation has been implied: associated with each element, x, there is a second element, $2x + 1$. Thus, a relation is often defined as a set of ordered pairs. The domain of the relation is the set of all first components of the ordered pairs. The range of the relation is the set of all second components.

A relation may be specified by actually listing the ordered pairs if the set is finite.

Example 1:

student	mark
Mary	87
Joe	82
Peter	95
Randy	76
Alice	68
Mark	82
Linda	92

The domain is {Mary, Joe, Peter, Randy, Alice, Mark, Linda} and the range is {87, 82, 95, 76, 68, 92}.

A relation is frequently specified by a rule which makes it possible to determine the ordered pairs. Students should be familiar with a variety of ways of expressing such a rule. Some possibilities follow:

$$F = \left\{ (x, y) \mid y = \frac{1}{\sqrt{x - 1}} \right\}$$

$$F: x \rightarrow \frac{1}{\sqrt{x - 1}}$$

$$F = \left\{ \left(x, \frac{1}{\sqrt{x - 1}} \right) \right\}$$

$$x \xrightarrow{F} \frac{1}{\sqrt{x - 1}}$$

$$F(x) = \frac{1}{\sqrt{x - 1}}$$

If not specified, the universe is generally understood to be the set of real numbers. The domain of the relation F, shown above, is the set of all reals greater than 1, i.e., $\{x \mid x > 1\}$, and the range is the set of all positive reals, i.e., $\{y \mid y > 0\}$.

A function is a special kind of relation in which each element of the domain corresponds to exactly one element of the range. In other words, a function is a relation in which no two ordered pairs have the same first element. F, shown above, is a function.

A function is a one-to-one function if and only if each second element corresponds to a unique first element. The set of ordered pairs obtained by interchanging the first and second elements of each pair of a one-to-one function, F, is the inverse function, F^{-1}.

The idea of inverse is familiar to students in different settings. They have seen that using the inverse "will undo what has been done:"

-3 is the additive inverse of 3 and $4 + 3 = 7$ and $7 + (-3) = 4$

2 is the multiplicative inverse of $\frac{1}{2}$ and $6 \times \frac{1}{2} = 3$ and $3 \times 2 = 6$

It seems reasonable, then, to consider f^{-1} as the inverse of f if, for all x, $x \xrightarrow{f} y$ and $y \xrightarrow{f^{-1}} x$

Example: If $f(x) = 2x + 1$, then $f^{-1}(x) = \frac{x - 1}{2}$

$$x \xrightarrow{f} 2x + 1 \text{ and } 2x + 1 \xrightarrow{f^{-1}} x$$

The result of interchanging the first and second elements of each pair of a function which is not one-to-one is not a function and, therefore, is not an inverse. Consider, for example,

the function $F(x) = y = x^2$. If x and y are interchanged, $x = y^2$ or $y = \pm\sqrt{x}$. Call this relation G. Then,

$x \xrightarrow{F} x^2$ and $x^2 \xrightarrow{G} \pm x$. Starting with x, the result of applying F and then G can be either x or $-x$. Therefore G did not necessarily "undo what had been done" by F and is not the inverse of F.

A common method of producing an inverse function when the original function is not one-to-one is to restrict the domain of the given function to make it one-to-one.

Example 1: $F: x \longrightarrow x^2$ Let the domain of F be the set of non-negative real numbers.

$F^{-1}: x \longrightarrow \sqrt{x}$ F^{-1} is the inverse of F. The domain of F^{-1} is the range of F, the set of non-negative real numbers.

When logarithms are introduced, there is the following example.

Example 2: $F: x \longrightarrow 2^x$

$F^{-1}: x \longrightarrow \log_2 x$

Since F is one-to-one, F^{-1} is the inverse function. Its domain is the range of F, the positive reals.

After the introduction of the trigonometry functions, there is the following example.

Example 3: $G: x \longrightarrow \sin x$

$H: x \longrightarrow \text{arc sin } x$

Since G is not one-to-one, H cannot be an inverse of G, unless the domain of G is restricted.

$G: x \longrightarrow \sin x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$G^{-1}: x \longrightarrow \text{Arc sin } x$

The range of G^{-1} is $\{-\frac{\pi}{2} \leq \text{Arc sin } x \leq \frac{\pi}{2}\}$

B. Two important types of functions are the exponential functions and their inverses, the logarithmic functions. Students have already encountered algebraic expressions involving exponential notation and have studied the rules for operating with them. These can be briefly reconsidered before introducing the usual definitions for a^n (n rational).

$$a^1 = a;\ a^n = \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text{ factors}};\ a^0 = 1,\ a \neq 0$$

$$a^{-n} = 1/a^n$$

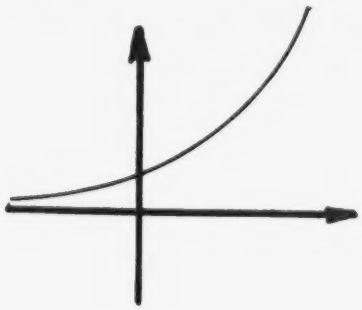
$$a^{1/n} = \sqrt[n]{a}$$

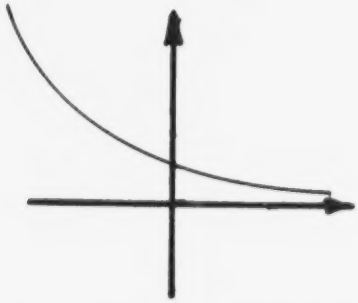
$$a^{m/n} = (a^{1/n})^m,\ a > 0$$

} m, n natural numbers

Since complex numbers were introduced in section 1, $a^{1/n}$ names a number with which students are familiar even in the case $a < 0$. However, $(a^{1/n})^m = (a^m)^{1/n}$ requires that $a > 0$. (For example, $[(-4)^{\frac{1}{2}}]^3 \neq [(-4)^3]^{\frac{1}{2}}$). Teachers may wish to justify these definitions as they are presented by appeal to the basic laws of exponents. Teachers should include in their development a discussion of scientific notation and the use of exponents to express very large or very small numbers.

An exponential function may be defined as $\{(x, y) \mid y = b^x\}$ where b is a positive real number different from 1. It is suggested that students be asked to sketch the graphs of several exponential functions since their nature is often better understood in this way. Below are sketches of $y = 2^x$ and $y = (\frac{1}{2})^x$. Of course, the graph of $y = a^x$, for any nonzero a will contain the point (0, 1).





The calculation of various values in the interval from -1 to 1 might be emphasized. It is suggested that teachers discuss the meaning of the expression "exponential growth" and its application to investigations in the physical and social sciences.

If $y = a^x$ then $x = \log_a y$. If $f(x) = a^x$, then $g(x) = \log_a x$ is the inverse of f. Students should graph some simple logarithmic functions such as $y = \log_2 x$ and compare them with the graphs of their inverses. It is hoped that the emphasis on the functional and graphic aspects of the exponential and logarithmic functions will be helpful to students in their future study of mathematics.

C. Logarithms to the base ten, or common logarithms, have historical importance as computational aids and as the basis of the slide rule. While electronic calculating devices are replacing such logarithmic computation, students should understand the principles of these computations and have some experience with them.

III. Circular Functions

A. In previous courses, the student has defined angle, congruent angles, and degree measure of an angle. It is useful to define degree measure of a minor arc of a circle as equal to that of the central angle whose rays contain the endpoints of the arc. The wording of this concept and those which follow from it should be consistent with that used by students in the study of angles in *Course II*. The definitions of chord, secant, and tangent should be introduced or reviewed and the measures of the angles formed when these line segments intersect should be related to the measures of their intercepted arcs.

Using these results of arc and angle measure, it is possible to prove theorems related to the circle which make use of similar triangle relationships. The proofs of theorems such as the following may be useful in themselves and in reviewing the concept of similarity.

- If two chords intersect within a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.
- Two tangent segments drawn to a circle from an external point are congruent.
- If a tangent and a secant are drawn to a circle from an external point, the measure of the tangent segment is the mean proportional between the measure of the secant and the measure of its external segment.

Most of the previous work with angles has been concerned with a static concept of angles in convex polygonal figures and thus limited to angles whose measures are less than that of a straight angle. But it is also useful to consider a dynamic interpretation of angle in terms of rotation. Such a definition makes it possible for any real number to be the measure of an angle.

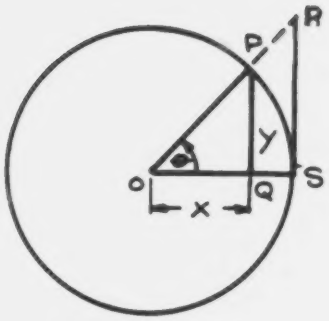
While degree measure is convenient in relating angle or arc size to that of a complete rotation, it is often more important to use a measure which relates angle or arc measure to linear measure. If a central angle of a circle intercepts an arc equal in length to the radius of the circle, then the measure of the angle is one radian. In particular, in a circle of radius one, a central angle of one radian intercepts an arc of one unit.

In this same circle, a central angle of n radians intercepts an arc of n units. Since the length of arc of one complete rotation is the circumference of the circle, $2\pi r$, or for a circle of radius one, 2π, a rotation of measure 360° is equivalent to that of 2π radians. Students should be able to express angle measure in terms of either degree or radian measure.

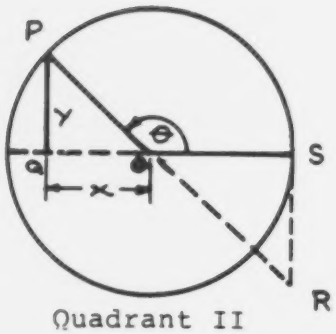
B. In the Cartesian coordinate system, a circle with center at the origin and radius one is designated by $\{(x, y) \; x^2 + y^2 = 1\}$. If a ray coincident with the positive x-axis with endpoint at the origin is rotated through θ radians, a point on that ray one unit from the origin traces out a path on the unit circle of length θ. The final position of this ray, called the terminal side, determines both an angle, and an arc of the unit circle, of measure θ. Thus (1, 0) represents the zero position for measuring arc length along the unit circle. Motion in the counterclockwise direction is considered positive and motion in the clockwise direction considered negative. For each real number, θ, which represents a length of path on the unit circle, there is a point with coordinates (x, y) such that $x^2 + y^2 = 1$. The sine function is defined as the set of ordered pairs $\{(\theta, y)\}$ and the cosine function is defined as the set of ordered pairs $\{(\theta, x)\}$. That is, $\sin \theta = y$ means that $\sin \theta$ is the vertical distance from the x-axis to the endpoint of the arc of length θ on the unit circle and $\cos \theta = x$ means that $\cos \theta$ is the horizontal distance from the y-axis to the endpoint of that arc. The domain of each function is the set of real numbers and the range is the set of numbers from -1 to 1 inclusive.

Using these same definitions of x, y, and θ, $\tan \theta$ is defined to be $\frac{y}{x}$, $\sec \theta$ to be $\frac{1}{x}$, $\cot \theta$ to be $\frac{x}{y}$, and $\csc \theta$ to be $\frac{1}{y}$ for all x, y for which the functions are defined.

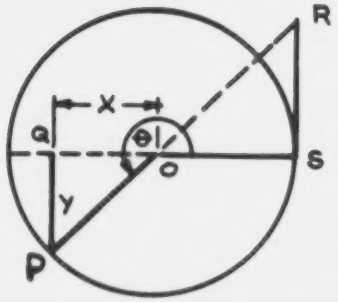
It is convenient to represent the function values for the sine, cosine, and tangent functions as the lengths of line segments related to the unit circle.



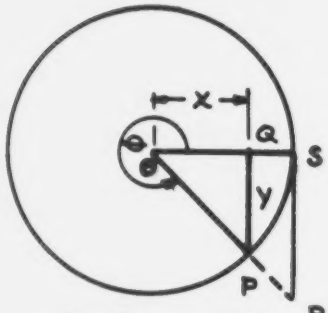
Quadrant I



Quadrant II



Quadrant III



Quadrant IV

In these diagrams, $\sin \theta = PQ$, $\cos \theta = OQ$. By definition, $\tan \theta = \frac{y}{x}$, and, since $\Delta OPQ \sim \Delta ORS$, $\frac{y}{x} = \frac{RS}{1}$ since these are the measures of corresponding parts of similar triangles. Therefore $\tan \theta = RS$. Students should be familiar with common angles of measure $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians, and their multiples, and will find it convenient to know or be able to determine the exact values of the sines, cosines, and tangents for these angles. Students should also be able to express the function value of any angle in terms of a function value of an acute angle.

The notation $\{(\theta, y)\}$ and $\{(\theta, x)\}$ used to define the sine and cosine functions is not the usual notation used in graphing and the student should understand that the change to the notation $y = \sin x$ and $y = \cos x$ in no way changes the concept involved. The significance of a and b in the expressions $y = a \sin bx$ and $y = a \cos bx$ should be considered and utilized in graphing these functions for simple positive values of a and b. The notion of <u>period</u> and <u>amplitude</u> should be introduced. The measure of the angle, x, should be expressed in radians and the scale on the x-axis made comparable to that on the y-axis. The graph of $y = \tan x$ should also be considered.

Consideration should be given to the inverse trigonometric functions, $y = \text{Arc sin } x$, $y = \text{Arc cos } x$, and $y = \text{Arc tan } x$.

C. An immediate consequence of the definitions of the tangent, cotangent, secant, and cosecant functions are the identities:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \qquad \tan \theta = \frac{1}{\cot \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

Since the sine and cosine functions have been defined in terms of the unit circle, $x^2 + y^2 = 1$, it follows that:

$$\cos^2\theta + \sin^2\theta = 1$$

By dividing both sides of this equation by $\cos^2\theta$ or by $\sin^2\theta$, it follows that:

$$1 + \tan^2\theta = \sec^2\theta \text{ if } \cos \theta \neq 0$$

$$\cot^2\theta + 1 = \csc^2\theta \text{ if } \sin \theta \neq 0$$

These eight basic identities should be used to prove other trigonometric identities. Since many trigonometric identities involve possibly troublesome algebraic expressions such as complex fractions, the teacher may need to introduce or review the principles involved in working with such expressions as a part of this section. Attention should be given to equations involving trigonometric functions, including quadratics, some of whose solutions require use of the quadratic formula.

D. If P is any point on the unit circle, $x^2 + y^2 = 1$, such that the length of arc from the point (1, 0) to P is θ, then the coordinates of P are $(\cos \theta, \sin \theta)$. If P' is any point on the ray OP, then $\Delta OPQ \sim \Delta OP'Q'$. Therefore, $\frac{PQ}{OP} = \frac{P'Q'}{OP'}$

Since $PQ = \sin \theta$ and if $OP' = a$,

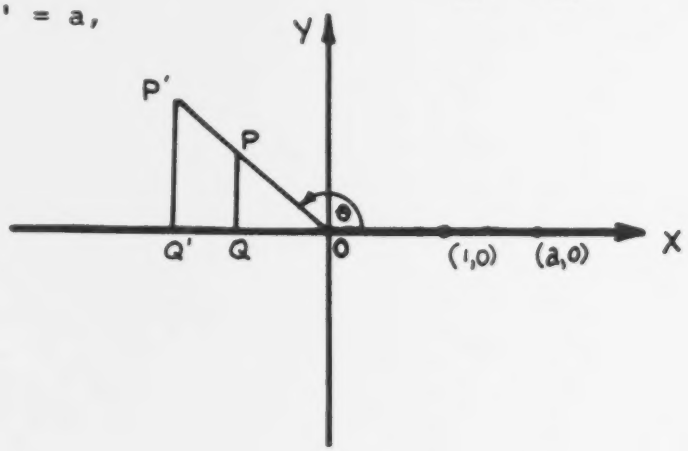
$$\frac{\sin \theta}{1} = \frac{P'Q'}{a}$$

$$P'Q' = a \sin \theta$$

and

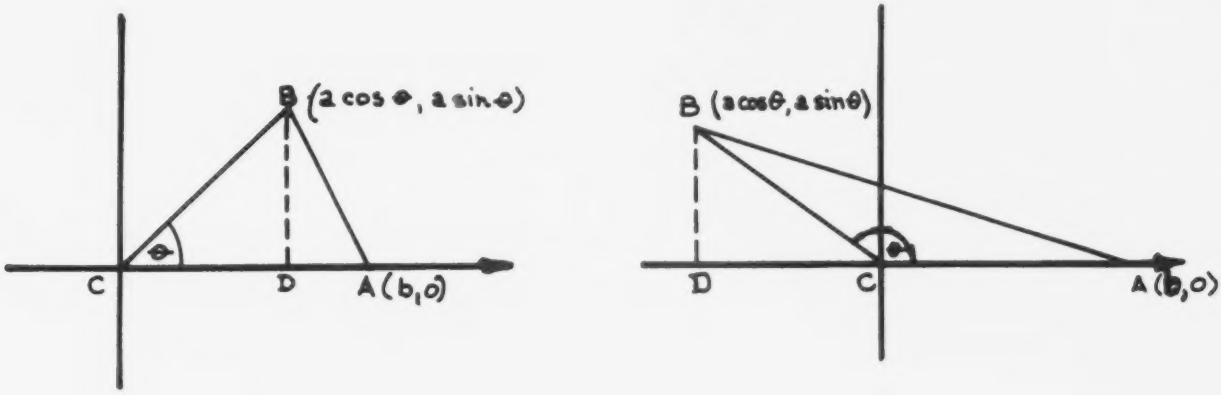
$$\frac{\cos \theta}{1} = \frac{OQ'}{a}$$

$$OQ' = a \cos \theta$$



Therefore the coordinates of P' are (a cos θ, a sin θ).

Any triangle may be associated with the coordinate system so that one vertex of the triangle is at the origin and one side of the triangle lies along the x-axis. Then the distance formula makes it possible to determine the length of one side in terms of the lengths of the other two sides and the measure of the angle between them.



$$c^2 = (b - a\cos\theta)^2 + (a\sin\theta)^2$$

$$c^2 = a^2\cos^2\theta - 2ab\cos\theta + b^2 + a^2\sin^2\theta$$

$$c^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2 - 2ab\cos\theta$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

or,

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{(Law of Cosines)}$$

Using these same triangles,

$$\text{Area of } \Delta ABC = \tfrac{1}{2}(CA)(BD) \text{ or,}$$

$$\text{Area of } \Delta ABC = \tfrac{1}{2}(b)(a\sin C)$$

By choosing the coordinate system appropriately, three expressions for area may be obtained:

$$\text{Area of } \Delta ABC = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}bc\sin A$$

If each member of this equality is divided by ½abc, the result is the Law of Sines:

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

The student should have some experience in solving triangles and in working with physical and other applications of this material, such as those concerning resolution of forces.

The rules derived above apply to all triangles and therefore to right triangles. However, for problems involving right triangles — lighthouses, shadows, angles of elevation, etc.. the teacher may wish to introduce the special relationship of right triangles,

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

which can be derived by applying the definitions of sine, cosine, and tangent to a first quadrant angle.

The teacher may wish to derive the formula for cos (A — B) by using the Law of Cosines. Once this formula is established, the formulas for sines and cosines of the sum or difference of two angles, double angles, or half angles can be derived. Students should understand the interrelationship of the trigonometic functions and be able to make use of the formulas in various applications. The solution of equations such as $\cos 2x + \cos x + 1 = 0$ should be included.

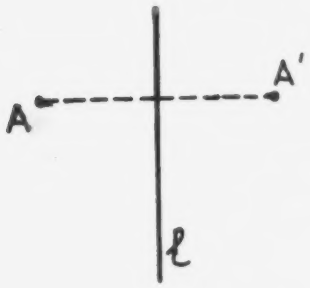
Teachers should note that a list of formulas appears in the *Course III* Regents Examination. The list is given in Appendix A of this publication. It is suggested that teachers make such a list available to their students throughout the year.

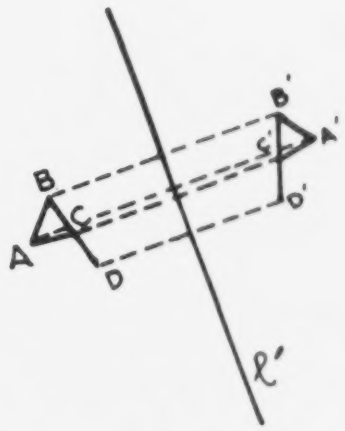
IV. Transformation Geometry

Many rewarding presentations are possible in the area of transformation geometry. This is a topic which can bring together many different strands of the discipline of mathematics. Depending on the abilities and interests of their students, some teachers may wish to stress a manipulative or discovery approach, while others will choose a more deductive presentation. Different emphases are possible with regard to the group-theoretical ideas inherent in the topic, as well as with regard to the ideas of similarity and congruence.

A. A transformation of the plane is a one-to-one mapping of the plane onto itself. The first example of a transformation in this section is the line reflection.

B. A reflection in line ℓ (suggested notation: r_ℓ) is a mapping which takes every point A of the plane into a point A' such that $\overline{AA'}$ is perpendicular to ℓ and the distance from A to ℓ equals the distance from ℓ to A'.



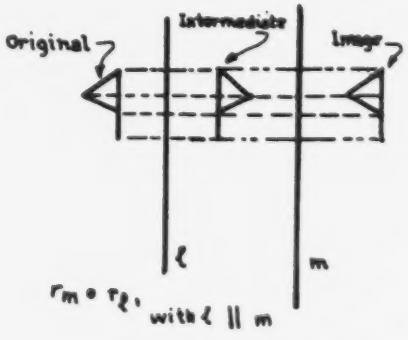


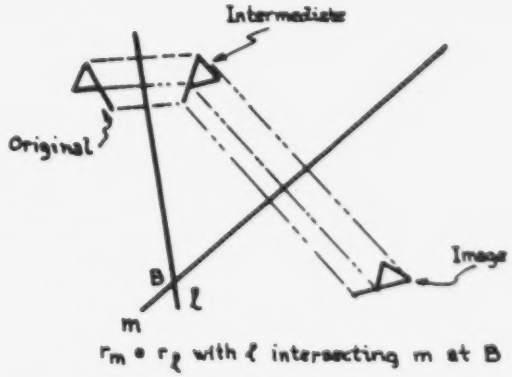
Flag figure A'B'C'D' is the image of ABCD under $r_{\ell'}$, reflection in line ℓ'.

Note that all points on line ℓ are mapped into themselves.

The line reflection preserves distance (i.e., AB = A'B') as well as betweenness (C between D and B yields C' between D' and B'), angle measure, and parallelism. The line reflection does not, however, preserve order (orientation); e.g., A, B, C are oriented clockwise, whereas A', B', C' are oriented counterclockwise.

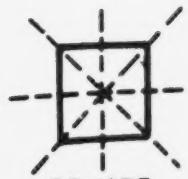
It is possible to consider the composition of two or more line reflections. Below are two illustrations. The notation $r_m \circ r_l$ denotes reflection in line l followed by reflection in line m.



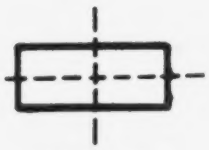


Note that the square of a line reflection (i.e., the composition of a line reflection with itself) takes every point of the plane back into itself and is, therefore, an identity mapping.

A figure which is its own image under some line reflection is said to have line-symmetry, or to be symmetric in, or about, the line. A figure may have more than one such symmetry.

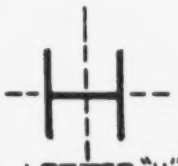


SQUARE
4 Lines of Symmetry



RECTANGLE
2 Lines of Symmetry

LETTER "A"
1 Line of Symmetry



LETTER "H"
2 Lines of Symmetry

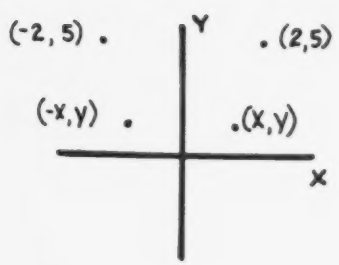
LETTER "R"
No Lines

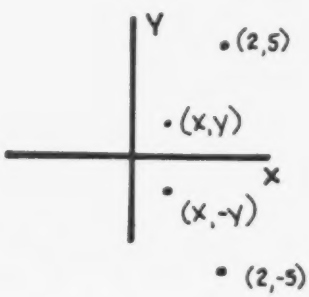
LETTER "O"
Infinitely many lines of Symmetry

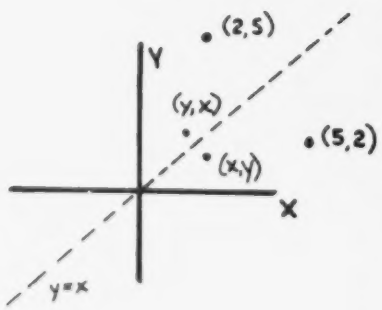
A transformation which preserves distance is called an <u>isometry</u>. An isometry which preserves order is called a <u>direct isometry</u>. Thus, the line reflection is an example of an isometry, but not a direct isometry.

Consider the special case of the coordinatized x,y plane. Under a reflection in the y-axis, the image of point P with coordinates (x, y) is point P' with coordinates (− x, y). Or, using function notation: $r_{y\text{-axis}}$: (x, y) ⟶ (− x, y). Similarly,

$r_{x\text{-axis}}$: (x, y) ⟶ (x, − y) and $r_{x = y}$: (x, y) ⟶ (y, x)

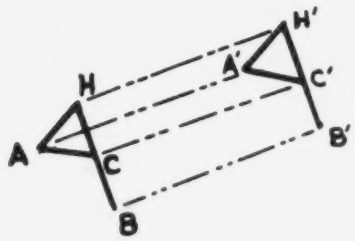






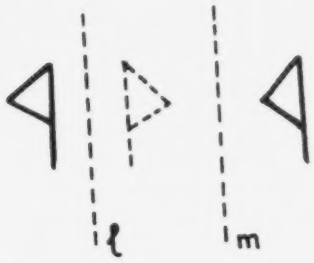
Students should be encouraged to discover these ideas through their own investigation. A good challenge for the more interested students is to try to find a similar rule for a reflection, say, in the line x = 2.

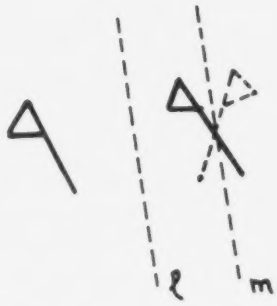
C. A translation is a mapping which assigns to every point in the plane another point in the plane such that every image point is located the same distance away, and in the same direction, from its pre-image, as every other. Or, more simply, a translation "moves" every point in the plane the same distance in the same direction.



All the points in flag-figure AHCB are "moved" (with every other point in the plane) equal distances in the same direction, in this case to A'H'C'B'.

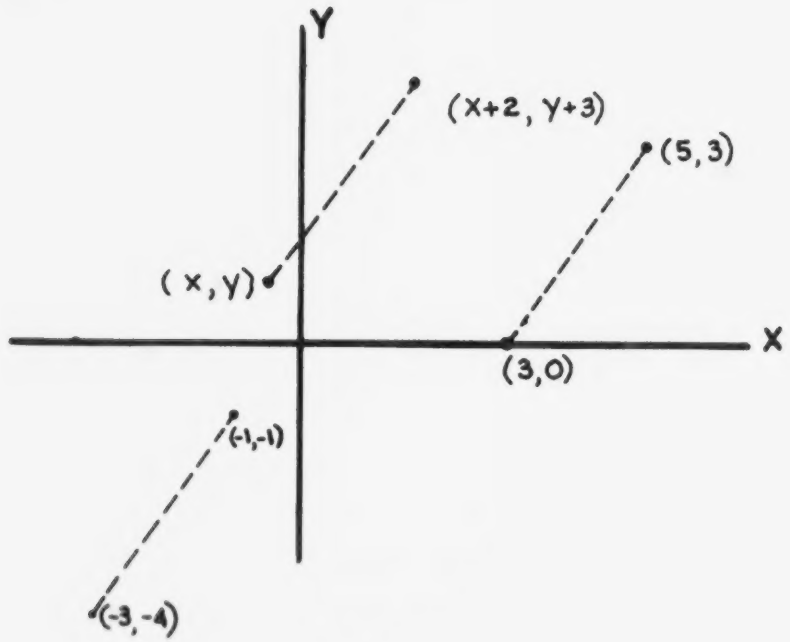
A translation can also be considered as the (non-unique) product of two line reflections in parallel lines (axes), as these diagrams illustrate:





Translation can be defined in this manner, although this approach is not suggested at this level. Note that since they preserve distance and order, translations are direct isometries.

With respect to a coordinatized x,y-plane, the translation which "moves" a point (x, y) a units in the x-direction, and b units in the y-direction (suggested notation: $T_{a,\ b}$) has the rule, $T_{a,\ b}$: $(x, y) \longrightarrow (x + a, y + b)$.

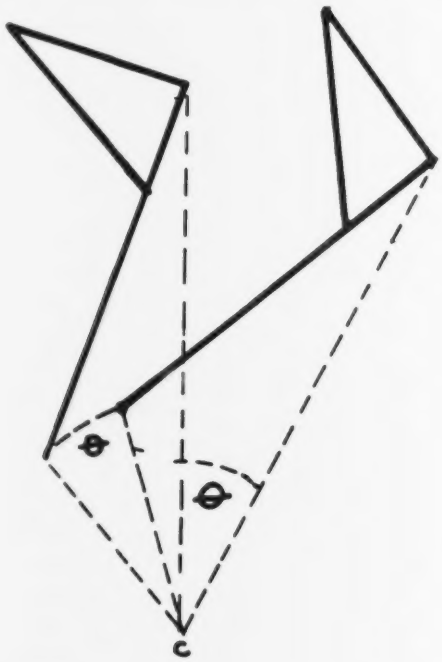


Images of points under translation $T_{2,\ 3}$

A translation which keeps every point where it is (i.e., maps every point onto itself; or, moves every point zero units in any direction) is the identity transformation. It can be called $T_{0,\ 0}$. Furthermore, for every translation $T_{a,\ b}$ there exists another translation $T_{-a,\ -b}$ which will take every image-point under $T_{a,\ b}$ back to where it started. Thus the product, or composition, $T_{-a,\ -b} \cdot T_{a,\ b} = T_{0,\ 0}$; i.e., $T_{a,\ b}$ and $T_{-a,\ -b}$ are inverses of each other.

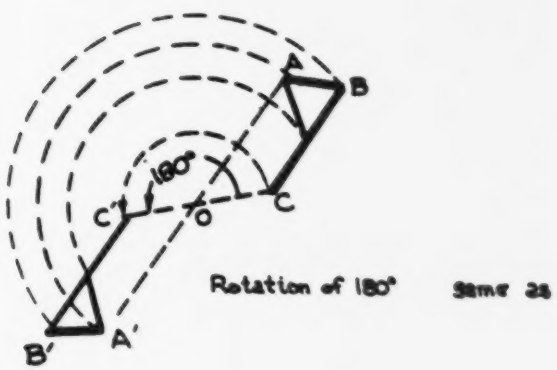
The idea of translational symmetry is probably more difficult to grasp than line or rotational symmetry. For this reason, many teachers may wish not to introduce it except to their more talented students. A figure has translational symmetry if there exists a translation that maps the figure onto itself. Note that no figure of finite proportions can have translational symmetry. An example of translational symmetry is a wallpaper or other "never ending" pattern. Other examples are the graphs of the trigonometric functions.

D. A rotation R about center, c, and through an angle of θ (notation: $R_{c,\theta}$ or $Rot_{c,\theta}$) is a mapping which "moves" all points in the plane through an angle of θ with point c fixed at the vertex of the angle. (If θ is positive, the motion is counterclockwise. If θ is negative, the motion is clockwise.)

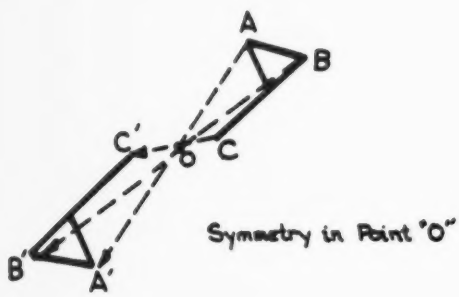


The center of the rotation is mapped to itself.

A rotation of 180 degress about a point P is sometimes called a <u>half-turn</u> or a <u>reflection in point P</u> (point-reflection).

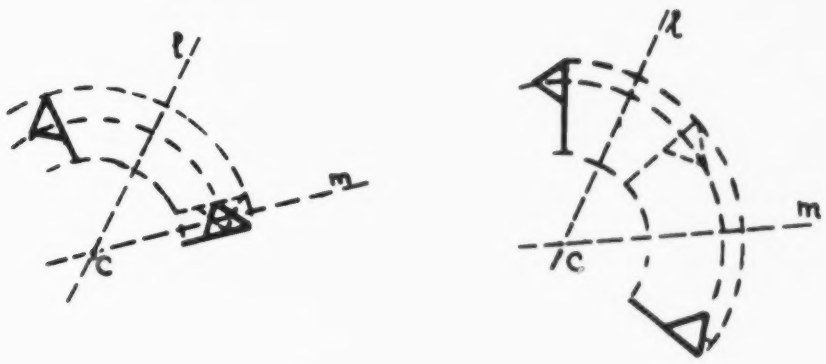


Rotation of 180° same as

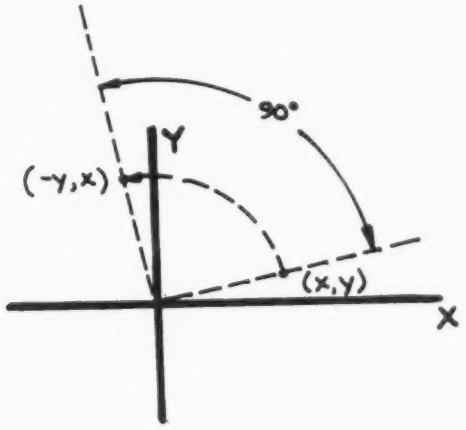


Symmetry in Point "O"

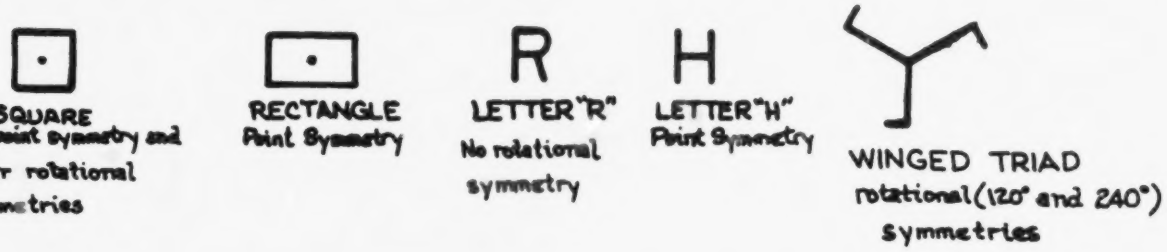
Note that a rotation about point C can be considered to be (and is often defined to be) the product of two line reflections whose axes intersect at C, as these diagrams illustrate:



Since rotations preserve distance and order, they are direct isometries. Discovery of rules for certain special rotations about the origin (90°, 180°, 270°, or -90°) is within the range of most students' abilities. For example, if $\theta = 90$, R_{90}: $(x, y) \rightarrow (-y, x)$ as this illustration shows. Note: If the center of rotation is the origin of the coordinate plane, then this is understood by writing simply R_{θ}.

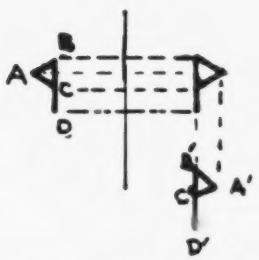


A figure has rotational symmetry if there exists a rotation (other than the identity) which maps it onto itself. If the figure has the 180°-rotational symmetry, it may be said to have point-symmetry. (Most crossword puzzles, for example, are point symmetric.)



The set of rotations of a plane about a fixed point forms a group under composition. This group has several interesting subgroups; for example, the set of rotations through 30k degrees, where k is an integer.

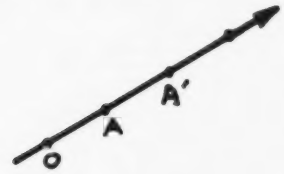
E. Thus far, three types of transformation have been discussed: reflections, translations, and rotations, all of which are isometries. It can be shown that every isometry is either one of these, or a glide-reflection. A glide-reflection is the composition of a line reflection and a translation in a direction parallel to the line of reflection.



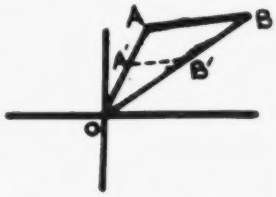
Flag figure ABCD mapped to figure A'B'C'D' under glide-reflection is shown to the left.

A glide-reflection can be seen to be the product of three line reflections (one for the reflection, two for the translation). Since translations and rotations are each equivalent to two line reflections, it can be seen that every isometry can be expressed as the composition of no more than three line reflections. This interesting result is often called the Three Line Reflection Theorem. Some teachers may wish to have their students "test it out" by "finding the reflections."

Probably most teachers would want to present an example of a transformation which is not an isometry. Perhaps the best example of such a mapping is the dilation. A dilation with center at the origin of a coordinate plane and scale factor a (notation: D_a) assigns to every point A another point A' on ray $\overrightarrow{OA}$ such that OA' = a(OA). Point 0 is mapped onto itself.



A mapped to A' under dilation D_2



$\Delta OAB \rightarrow \Delta OA'B'$ under $D_{\frac{1}{2}}$

$B \rightarrow B'$ under D_{-1}

Since distances are not preserved (for example, $OA \neq OA'$), dilations are not isometries. One exception is D_1 which maps the plane onto itself and is therefore the identity; the other is D_{-1} which is equivalent to a point reflection since it also preserves distance.

F. A study of transformation geometry can provide many applications to group theory. For example, since:

the composition of any two translations is a translation

it can be demonstrated that the composition of translations is associative

there exists an identity translation $(T_{0,0})$

every translation has an inverse

The set of all translations, together with the operation composition, constitutes a group. That the group is also commutative is a useful investigation.

The set of rotations of a plane about a fixed point also forms a group under composition. This group has several interesting subgroups; for example, the set of rotations through 30k degrees, where k is an integer.

It can also be shown that the set of all isometries forms a group under composition and that the set of direct isometries forms a subgroup of this group.

These ideas may be pursued as an optional topic with more talented students.

The study of transformation geometry provides alternate definitions of congruence and similarity. Two figures are congruent if there exists an isometry under which one is the image of the other. Two figures are similar if there exists an isometry and a dilation under the composition of which one figure is mapped to the other. Teachers are encouraged to discuss fully this notion of congruence and to compare and contrast it with the concepts students have already met in their study of Euclidean geometry. Some teachers may want to have their classes develop isometry-oriented proofs for theorems, or other propositions, previously justified in other ways. Hopefully, students will come to agree that looking at a mathematical question from more than one point of view, or with more than one set of assumptions and definitions, is a valuable and enjoyable experience for mathematics students.

V. Probability and Statistics

A. B. In previous courses, students have studied simple probability problems such as finding the probability of a "6" on a single toss of a fair die. Such problems may be reviewed along with other aspects of this material from *Course I* and *Course II* such as the counting principle and elementary combinatorics. This study may now be extended to situations involving the probability of exactly, at least, or at most r successes in n trials.

Example 1: Find the probability of rolling one "6" in six tosses of a fair die.

If the problem were, "Find the probability of the event E of rolling a fair die six times and obtaining 6 on the first roll and some other number on each of the others," then

$P(E) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5$ since the probability of a 6 is $\frac{1}{6}$ and the probability of some other number is $\frac{5}{6}$. But in the original problem, the 6 can be either on the first toss, the second, and so on, up to the sixth. Since the one required 6 can occur in any one of six places, and since the probability of its occurrence in any one of these places is the same, it can be concluded that:

$$P(\text{exactly one 6 in six tosses}) = 6 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = \frac{3125}{7776} \approx .402$$

Students may compare this with empirical results.

To find the probability of obtaining exactly two 6's in six tosses, consider initially the probability that the first two tosses, only, will be 6's. This is given by:

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 \approx .0134$$

But the 6's in the problem do not have to be the first two. Two things (for example, two 6's) can be distributed among six places in ${}_6C_2$ ways. Therefore, the result is:

P(exactly two 6's in six tosses) =

$${}_6C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 = 15 \cdot \frac{1}{36} \cdot \frac{625}{1296} \approx .201$$

It follows that the probability of obtaining exactly r 6's $(0 < r < 6)$ in six tosses is ${}_6C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{6-r}$ and of obtaining r 6's in n tosses is ${}_nC_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{n-r}$.

Finally, it follows that in the case of an experiment where there are exactly two outcomes, (6 or not 6; heads or tails; American League or National League; etc.), with probability of "success" p and probability of "failure" $q = 1 - p$, that the probability of r successes in n trials is $\binom{n}{r} p^r q^{n-r}$.

Experiments with exactly two outcomes are called Bernoulli experiments (after James Bernoulli, 1654-1705). The individual "tosses" discussed here are examples of independent trials of a Bernoulli experiment.

Example 2: The probability, p, of obtaining at least eight heads in ten tosses of a fair coin is equal to the sum of the probabilities of obtaining exactly eight, exactly nine, and exactly ten heads.

$$P = \binom{10}{8}\left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + \binom{10}{10}\left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

Example 3: If the coin is "loaded" so that P (heads) = $\frac{1}{3}$, P (tails) = $\frac{2}{3}$, then the probability of r heads in n tosses is $\binom{n}{r}\left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{n-r}$

C. With this background, the student is prepared to move on to a study of the Binomial Theorem. There are a variety of approaches to this theorem, but it is hoped that whatever approach is used, students will come to see the theorem as an integral part of significant mathematical situations rather than as an end in itself. It is also hoped that whatever notation will be involved in expressing the theorem will have become so reasonable and ordinary to most students, that it will not act as a barrier to understanding.

Students may be encouraged to look for patterns among coefficients of $(a + b)^n$ for $n = 1, 2, 3, \ldots$ and may discover or be led to observe that these coefficients are the entries in Pascal's Triangle as well as different values of ${}_nC_r$ for various n and r.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

etc.

Students might now be asked to reconsider the problem of tossing a coin - not necessarily a fair one - with probability of heads p and probability of tails q. If the coin is tossed n times, a chart, similar to the one on the following page, might be developed for different values of n.

n	Possible Outcomes	Probability of Outcomes	Probability of r heads $0 \leq r \leq n$	
1	(H)	p	$r = 1$ $P(1 \text{ head}) = p$	$p + q = 1$
	(T)	q	$r = 0$ $P(0 \text{ heads}) = q$	
2	(H,H)	$pp = p^2$	$r = 2$ $P(2 \text{ heads}) = p^2$	$p^2 + 2pq + q^2 = 1$
	(H,T)	pq	$r = 1$ $P(1 \text{ head}) = 2pq$	
	(T,H)	$qp = pq$		
	(T,T)	$qq = q^2$	$r = 0$ $P(0 \text{ heads}) = q^2$	
3	(H,H,H)	p^3	$r = 3$ $P(3 \text{ heads}) = p^3$	$p^3 + 3p^2q + 3pq^2 + q^3 = 1$
	(H,H,T)	$ppq = p^2q$		
	(H,T,H)	$pqp = p^2q$	$r = 2$ $P(2 \text{ heads}) = 3p^2q$	
	(T,H,H)	$qpp = p^2q$		
	(T,T,H)	$qqp = pq^2$		
	(T,H,T)	$qpq = pq^2$	$r = 1$ $P(1 \text{ head}) = 3pq^2$	
	(H,T,T)	$pqq = pq^2$		
	(T,T,T)	q^3	$r = 0$ $P(0 \text{ heads}) = q^3$	
4	(H,H,H,H)	p^4	$r = 4$ $P(4 \text{ heads}) = p^4$	$p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4 = 1$
	↓	↓	$r = 3$ $P(3 \text{ heads}) = 4p^3q$	
			$r = 2$ $P(2 \text{ heads}) = 6p^2q^2$	
			$r = 1$ $P(1 \text{ head}) = 4pq^3$	
	(T,T,T,T)	q^4	$r = 0$ $P(0 \text{ heads}) = q^4$	
5	ETC.			

The entries in the right column of the table suggest a statement of the Binomial Theorem:

$$(p + q)^n = \binom{n}{0} p^n + \binom{n}{1} p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \ldots + \binom{n}{n}q^n$$

Since p and q can represent any algebraic expressions, the formula can be used to do things like finding the expansion of $(\frac{1}{2}x - 2y)^5$.

Depending on the skills and interests of their classes, teachers may wish to continue with this topic through a more detailed discussion of the binomial distribution. For more capable students, transition into the realm of statistics could be, for example, a demonstration that for a binomial distribution (n trials, probability of success p) the mean is equal to the product np.

D. E. In *Course I*, students have discussed three statistical measures: the mean, median and mode. (See outline and notes for *Course I*.) While these measures, especially the mean, are very useful measures of central tendency, it is clear that for certain kinds of investigation, they are insufficient. For example, suppose that one thousand high school students are given some standardized test as part of a special research project, and suppose that the mean of the scores is 80. This might seem to be directly usable information in and of itself until it is realized that this mean of 80 could have been the result either of all students scoring approximately 80 on the test or as a result of half scoring 60 and half scoring 100. Obviously, this is quite different in terms of the research project. Teachers can easily make up many examples where the measures of central tendency already studied are insufficient sources of usable information about a set of numbers. Therefore, in addition to a measure of central tendency, some measure of the degree of deviation from that central tendency, or of dispersion, is necessary.

One of the simplest measures of dispersion is the range, which is the difference between the highest and lowest numbers in the set. Sometimes this measure is useful but since it depends on only two numbers, it is easy to see circumstances in which it would be inadequate. Some students might suggest that a measure of dispersion using all the numbers might be the sum of the deviations, $(\bar{x} - x_i)$, of each measure from the mean, $\bar{x}$, but of course, this sum turns out to be zero. However, if the sign of each difference is disregarded, then each of the differences would be nonnegative and their sum different from zero (except in the rare case when all the data are equal). The measure, in symbols, $\frac{1}{n}\sum_{i=1}^{n} |\bar{x} - x_i|$, is called the mean

absolute deviation. ($\sum$ - notation will be discussed later.) A disadvantage of this measure is its unwieldiness when large numbers of data are being considered.

A more widely used measure of dispersion is the variance:

$$\text{variance of a set of n numbers} = \frac{1}{n}\sum_{i=1}^{n}(\bar{x} - x_i)^2$$

Here each difference of a number from the mean is squared, thus assuring that the sum will be nonnegative. The sum of these squared deviations is divided by n, the total number of measures, to keep this measure at a manageable size.

An objection to the variance as a measure of dispersion is the fact that it does include an element of distortion stemming from the fact that before the division, each difference was squared. One further step leads to what is probably the most important and widely used measure of dispersion, the standard deviation, the square root of the variance:

$$\text{s.d.} = \sqrt{\frac{1}{n}\sum_{i-1}^{n}(\bar{x} - x_i)^2}$$

where n is the total number of measures and $\bar{x}$ is the mean. (Other commonly used symbols for standard deviation are σ, and s.) The variance of a set of measures is often symbolized by s^2 to correspond to the use of s for standard deviation. Students should be asked to work out several problems in computing standard deviation, some involving fairly large numbers of measurements.

F. Since it is very difficult to deal with certain statistical investigations, consisting as they generally do of large accumulations of data, without a system of notation to make things more manageable and efficient, $\sum$ - notation has been introduced in this outline. Teachers may find the discussion of $\sum$ - notation a useful introduction to this material on statistics.

The $\sum$ (Greek capital sigma) means, in these applications, "take the sum of all of the following" and is often used with some indexing letters i, j, etc. This discussion will be limited to one index at a time, usually i.

Example 1: $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$

Example 2: A certain survey has produced these fifteen measures: 7, 1, 7, 3, 10, 7, 0, 1, 3, 6, 4, 0, 7, 7, 3.

If we index them in this way:

$$7_1,\ 1_2,\ 7_3,\ 3_4,\ 10_5,\ 7_6,\ 0_7,\ 1_8,\ 3_9,\ 6_{10},\ 4_{11},\ 0_{12},\ 7_{13},\ 7_{14},\ 3_{15},$$

then $\sum_{i=1}^{15} x_i$ = the sum of these measures in the order given = 66. Often it is wise to order the measures first before indexing and taking their sum:

$$0_1,\ 0_2,\ 1_3,\ 1_4,\ 3_5,\ 3_6,\ 3_7,\ 4_8,\ 6_9,\ 7_{10},\ 7_{11},\ 7_{12},\ 7_{13},\ 7_{14},\ 10_{15}.$$

In some cases it might prove efficient to group the measures according to their frequency of occurrence. The table below does this for the set of measures given above.

index	measure	frequency
1	0	2
2	1	2
3	3	3
4	4	1
5	6	1
6	7	5
7	10	1

From this table, we see that the first measure, 0, occurs with a corresponding first frequency 2; that the second measure, 1, occurs with a corresponding second frequency 2; and so on. Then, if we use "f" to represent "frequency" we may index the seven different measures and their corresponding frequencies: for $i = 1$, $f_i = 2$ and $x_i = 0$; $i = 2$, $f_i = 2$ and $x_i = 1$; ...; for $i = 7$, $f_i = 1$ and $x_i = 10$. Then

$$\sum_{i=1}^{7} f_i\, x_i = 2(0) + 2(1) + 3(3) + 1(4) + 1(6) + 5(7) + 1(10) = 66.$$

Such an organization is clearly quite useful where large numbers of repeated data are being computed.

If the symbol $\bar{x}$ (read "x-bar") is used to represent the mean, $\sum_{i=1}^{n} (\bar{x} - x_i)$ represents the sum of the differences of n measure from their mean, while the summation $\sum_{i=1}^{n} (\bar{x} - x_i)^2$ is the sum of the squares of the difference of the measures from the mean. Of course, this could also be written

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 .$$

Sometimes, when working with a set of numbers whose limits everyone knows quite well, the limits of summation are omitted. Thus, in this case, the sum given immediately above might be written $\sum (x_i - \bar{x})^2$. Since notation is shorthand for an ordinary sum, the familiar properties of addition and multiplication — associativity, commutativity and distributivity — apply. Thus, if c is a constant, $\sum cx_i = c\sum x_i$ and, for example,

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i .$$

These statements can be readily demonstrated by writing out the sum, rearranging the terms, and resumming.

Example: Calculate the mean absolute deviation, variance (s^2), and standard deviation (s) of the following set of data:

9, 16, 9.5, 12, 12, 9, 16, 16, 17.5, 19, 16, 12, 13, 12, 13, 14, 13, 16, 19, 16

Solution: Of the n measures, there are k different ones (i.e., the n measures can be grouped into k classes) each occurring with its own frequency. In this case, n = 20, k = 8. The mean is 14.

i Index	x_i Measure	f_i Frequency	$\lvert\bar{x} - x_i\rvert$	$f_i\lvert\bar{x} - x_i\rvert$	$(\bar{x} - x_i)^2$	$f_i\,(\bar{x} - x_i)^2$
1	9	2	5	10	25	50
2	9.5	1	4.5	4.5	20.25	20.25
3	12	4	2	8	4	16
4	13	3	1	3	1	3
5	14	1	0	0	0	0
6	16	6	2	12	4	24
7	17.5	1	3.5	3.5	12.25	12.25
8	19	2	5	10	25	50
				$\sum_{i=1}^{8} f_i\lvert\bar{x}-x_i\rvert$		$\sum_{i=1}^{8} f_i(\bar{x}-x_i)^2$
				51		175.5

Mean absolute deviation = $\frac{51}{20} = 2.55$

Variance = $\frac{175.5}{20} = 8.775$

Standard deviation ≈ 2.96.

Of course, students will probably be asked to deal with problems involving far more than twenty measurements.

The computation of variance and standard deviation as defined requires computation of $\bar{x} - x_i$ for all measures and squaring those differences. It can be shown, however, that the following equality holds:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$ and more skillful or well-motivated groups may wish to consider this demonstration:

$$\frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{1}{n}\sum [x_i^2 - 2x_i\bar{x} + \bar{x}^2]$$

$$= \frac{1}{n}\left[\sum x_i^2 - 2\bar{x}\sum x_i + \sum \bar{x}^2\right]$$

$$= \frac{1}{n}\sum x_i^2 - \left(\frac{1}{n} \cdot 2 \cdot \bar{x} \cdot n\bar{x}\right) + \frac{1}{n} \cdot n\bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - 2\bar{x}^2 + \bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

This formulation is especially useful if the measures are such that a limited number of them occur with some frequency. For, using the fact that for n measures grouped into k classes, $\sum_{i=1}^{k} f_i x_i = \sum_{i=1}^{n} x_i$, it can be shown that $s^2 = \frac{1}{n}\sum_{i=1}^{k} f_i \cdot (x_i)^2 - \bar{x}^2$.

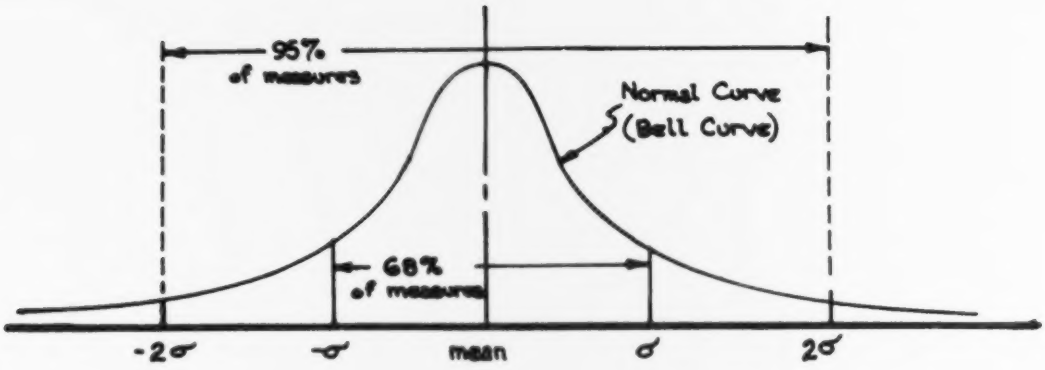
Example: For the set of measures given above, s^2 can be computed as

$$s^2 = \frac{1}{20}\sum_{i=1}^{8} f_i\,(x_i)^2 - 14^2$$

$$= \frac{4095.50}{20} - 196 = 8.775$$

Another useful problem for especially able students would be to show that for a binomial distribution, $s = \sqrt{npq}$.

All classes might find it useful to be introduced to some facts about "bell-curves," although few of them can be proved at this point. In a set of measures whose distribution is described by a bell curve — which formally is a normal distribution — 68 percent of the measures fall within one standard deviation of the mean and 95 percent within two standard deviations.



Teachers may find a discussion of such distributions, applied for example to examinations students have taken, a good illustration of statistical ideas.

VI. Optional Topics

As time permits, teachers may wish to explore the following topics:

A. Polar coordinates
B. Number theory
C. Mathematical induction
D. Sequences and series
E. Matrices
F. Synthetic substitution, remainder theorem, and factor theorem

APPENDIX A: FORMULAS

Pythagorean and Quotient Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A + 1 = \sec^2 A$$
$$\cot^2 A + 1 = \csc^2 A$$

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

Functions of the Sum of Two Angles

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Functions of the Difference of Two Angles

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Functions of the Double Angle

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 2 \cos^2 A - 1$$
$$\cos 2A = 1 - 2 \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Functions of the Half Angle

$$\sin \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \tfrac{1}{2}A = \pm \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Area of Triangle

$$K = \tfrac{1}{2} ab \sin C$$

Standard Deviation

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\bar{x} - x_i)^2}$$

APPENDIX B: NOTATIONS

Relations and Functions: F^{-1} the function which is the inverse of the function F

Inverse Trig. Functions: Arc sin, Arc cos, etc.

Composition of Mapping: f o g means g followed by f — "do g first"

Transformations: r_ℓ reflection in line ℓ

$R_{c,90}$ or $Rot_{c,90}$ rotation counter-clockwise about center c

$R_{c,-90}$ or $Rot_{c,-90}$ rotation clockwise about center c

R_{90} or Rot_{90} rotation about <u>origin</u> of 90°

$T_{a,b}$ translation a units horizontally, b units vertically

D_a dilation with center at origin and scale factor a of a coordinate plane

Prob./Stat./Combinatorics: Three notations, ${}_nC_r$, C(n,r) and $\binom{n}{r}$ appear in *Course II*

In *Course III*, teachers may use any of these, but only ${}_nC_r$ and $\binom{n}{r}$ appear in the outline.

$\bar{x}$ mean

$\bar{x} - x_i$ difference of a measure x_i from mean $\bar{x}$

s^2 variance

s standard deviation, but s.d., σ are also used

$\sum_{i=1}^{n}$ the sum of n numbers beginning with the first and ending with the nth.

APPENDIX C: SAMPLE REGENTS EXAMINATION QUESTIONS

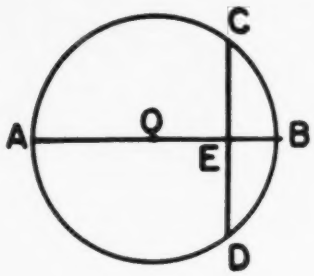
Sample Part I Questions

1 If $f(x) = 4x^{\frac{1}{2}}$, find $f(4)$.

2 Express the product of $(2 + \sqrt{-9})$ and $(3 - \sqrt{-16})$ in the form $a + bi$.

3 Express in *simplest form*: $\dfrac{n - \frac{1}{n}}{\frac{1 - n^2}{n}}$

4 In the accompanying diagram of circle O, diameter $\overline{AB} \perp \overline{CD}$, and $CD = 14$. Find CE.



5 Find $\tan\left(\text{Arc sin } \dfrac{\sqrt{2}}{2}\right)$.

6 In a circle, a central angle of 2 radians intercepts an arc of length 12 centimeters. Find the length of the radius in centimeters.

7 Express sin 230° as a function of a positive acute angle.

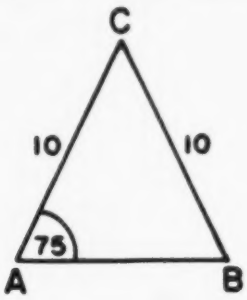
8 Express $\dfrac{7\pi}{10}$ radians in degree measure.

9 What is the image of the point $(2, -3)$ under a clockwise rotation of 90° ($R_{-90°}$) about the origin?

10 If $\log_9 81 = x$, find x.

11 Find the value of $\sum_{k=1}^{4} \sin \dfrac{k\pi}{2}$.

12 In the accompanying figure of $\triangle ABC$, $a = 10$, $b = 10$, and $m\angle A = 75$. Find the area of $\triangle ABC$.



13 Two tangents $\overline{PA}$ and $\overline{PB}$ are drawn to circle O from an external point P. If the measure of major arc $\overline{AB}$ is 250°, find the measure of $\angle P$.

14 Find the value of log 429.7 correct to *four decimal places*.

15 In $\triangle RST$, $\sin R = 0.6$, $\sin S = 0.4$, and side $s = 16$. Find the length of side r.

16 Find the value of x in the interval $90° < x < 180°$ which satisfies the equation $\cos x - 2 \cos x \sin x = 0$.

17 If $\cos x = \frac{3}{5}$, what is the positive value of $\sin \frac{1}{2}x$?

18 If the probability of winning a game is $\frac{1}{4}$, find the probability of winning *at least* 3 games out of 4.

Directions (19–35): For each question chosen, write on the separate answer sheet the *numeral* preceding the word or expression that best completes the statement or answers the question.

19 Which letter has vertical line symmetry?

(1) A (3) N
(2) S (4) D

20 Which graph has line symmetry with respect to the y-axis?

(1) $y = x$ (3) $y = \sin x$
(2) $y = x^2$ (4) $y = \tan x$

54

21 As angle x increases from 180° to 270°, the value of cos x will
(1) increase from 0 to 1
(2) increase from −1 to 0
(3) decrease from 0 to −1
(4) decrease from 1 to 0

22 The expression $\log \sqrt{xy}$ is equivalent to
(1) $2 \log x \log y$
(2) $2(\log x + \log y)$
(3) $\frac{1}{2}\log x \log y$
(4) $\frac{1}{2}(\log x + \log y)$

23 If, in $\triangle ABC$, $a = 5$, $b = 6$, and $c = 8$, then cos A is
(1) $-\frac{1}{20}$
(2) $\frac{11}{32}$
(3) $\frac{25}{32}$
(4) $\frac{53}{80}$

24 If $a = 4$, $b = 5$, and $m\angle A = 30$, the number of distinct triangles that may be constructed is
(1) 1
(2) 2
(3) 3
(4) 0

25 The maximum value of the function $y = 3 \sin 2x$ is
(1) π
(2) 2
(3) 3
(4) 2π

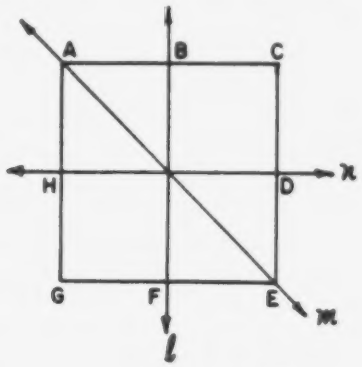
26 The expression $(\sin x - \cos x)^2$ is equivalent to
(1) 1
(2) $-\cos 2x$
(3) $1 - \sin 2x$
(4) $1 - \cos 2x$

27 The expression $\cos(90° + \theta)$ equals
(1) $\cos \theta$
(2) $-\cos \theta$
(3) $\sin \theta$
(4) $-\sin \theta$

28 The graph of $y = ax^2 + bx + c$ is tangent to the x-axis. The roots of the equation $ax^2 + bx + c = 0$ are
(1) real, rational, and unequal
(2) real, rational, and equal
(3) imaginary
(4) real, irrational, and unequal

29 If $f(x) = \cos x + \tan \frac{x}{3}$, then $f(\pi)$ is
(1) $\frac{\sqrt{3} + 3}{3}$
(2) $\frac{\sqrt{3} - 3}{3}$
(3) $\sqrt{3} + 1$
(4) $\sqrt{3} - 1$

30 In the accompanying figure, ℓ, m, and n are symmetry lines. What is $r_\ell \circ r_m(\overline{CD})$?



(1) $\overline{AB}$
(2) $\overline{CD}$
(3) $\overline{EF}$
(4) $\overline{GH}$

31 The expression $\frac{5}{2 - \sqrt{3}}$ is equivalent to
(1) $10 + 5\sqrt{3}$
(2) $-2 - \sqrt{3}$
(3) $-10 - 5\sqrt{3}$
(4) $2 + \sqrt{3}$

32 If $\cos x = -\frac{4}{5}$ and $\tan x > 0$, then $\angle x$ terminates in Quadrant
(1) I
(2) II
(3) III
(4) IV

33 Which is the 7th term in the expansion of $(2x - y)^7$?
(1) $7xy^6$
(2) $-7xy^6$
(3) $14xy^6$
(4) $-14xy^6$

34 Which correctly compares the mean and median of the set of data shown in the accompanying table?

x_i measure	f_i frequency
60	2
75	4
80	1
90	3

(1) The mean and median are equal.
(2) The mean exceeds the median by 2.
(3) The median exceeds the mean by 2.
(4) The mean exceeds the median by 2.5.

35 On a standardized test, the mean is 61 and the standard deviation is 3.2. Which score can be expected to occur *less than* 3% of the time?
(1) 50
(2) 56
(3) 62
(4) 65

Sample Part II Questions

1 *a* On the same set of axes, sketch the graphs of $y = \tan x$ and $y = \frac{1}{2}\cos x$, as x varies from 0 to 2π radians. [8]

b State the number of values of x in the interval $0 \le x \le 2\pi$ that satisfy the equation $\tan x = \frac{1}{2}\cos x$. [2]

2 *a* On the same set of axes, sketch the graphs of $y = \frac{1}{2}\sin x$ and $y = \cos 2x$ as x varies from 0 to 2π radians. [8]

b From the graphs made in part *a*, determine the number of values of x between 0 and 2π radians which satisfy the equation $\frac{1}{2}\sin x = \cos 2x$. [2]

3 *a* Find to the *nearest degree* all values of θ in the interval $0° \le \theta \le 360°$ which satisfy the equation $\sec^2 \theta = 4 - 2\tan \theta$. [6]

b Solve the equation $x^2 + 8x + 20 = 0$ and express its roots in the form $a + bi$. [4]

4 *a* Find all values of θ in the interval $0° \le \theta \le 360°$ which satisfy the equation $2\sin^2 \theta - 3\sin \theta + 1 = 0$. [4]

b For all values of θ for which the expressions are defined, prove that the following is an identity:

$$\frac{\cos \theta + \cot \theta}{\cos \theta \cot \theta} = \tan \theta + \sec \theta \quad [6]$$

5 *a* Solve the equation $x^2 - 2x + 5 = 0$ and express the roots in the form $a + bi$. [5]

b Using logarithms, solve for x to the *nearest tenth*.

$2^x = 5$ [5]

6 *a* Using logarithms, find $\sqrt[3]{0.692}$ to the *nearest hundredth*. [5]

b For all values of x for which the expressions are defined, prove the identity:

$$\frac{\tan \theta \csc^2 \theta}{1 + \tan^2 \theta} = \cot \theta \quad [5]$$

7 Find, to the *nearest tenth*, the standard deviation for the following set of data. [10]

x_i measure	f_i frequency
60	1
75	4
80	3
90	2

8 A high school football team scored the following number of points during the ten-game season:

19, 20, 21, 27, 29, 29, 34, 40, 40, 41

a What is the median? [1]

b What is the mean? [2]

c Find the standard deviation of these scores to the *nearest tenth*. [7]

9 Two forces of 30 pounds and 40 pounds act upon a body, forming an acute angle with each other. The angle between the resultant and the 30-pound force is 35° 10′. Find, to the *nearest ten minutes*, the angle between the two given forces. [10]

10 *a* Two consecutive sides of a parallelogram are 8 centimeters and 10 centimeters, respectively. If the length of the longer diagonal of the parallelogram is 14 centimeters, find the measure of the largest angle of the parallelogram to the *nearest degree*. [7]

b Using your answer to part *a*, find the area of the parallelogram to the *nearest square centimeter*. [3]

11 Given: F is the transformation $(x,y) \rightarrow (-y,-x)$
U is the transformation $(x,y) \rightarrow (x-2,y+4)$
N is the transformation $(x,y) \rightarrow (2x,2y)$

The coordinates of $\triangle ABC$ are $A(1,2)$, $B(4,0)$, and $C(3,-2)$.

a Sketch $\triangle ABC$ and its image $\triangle A'B'C'$ after the transformation F. [3]

b Sketch $\triangle A''B''C''$, the image of $\triangle A'B'C'$ after the transformation U. [3]

c Sketch $\triangle A'''B'''C'''$, the image of $\triangle A''B''C''$ after the transformation N. [3]

d Which transformation, F, U, or N, is a dilation? [1]

12 The coordinates of the endpoints of line segment $\overline{AB}$ are $A(4,1)$ and $B(5,4)$.

a Graph $\overline{AB}$. [2]

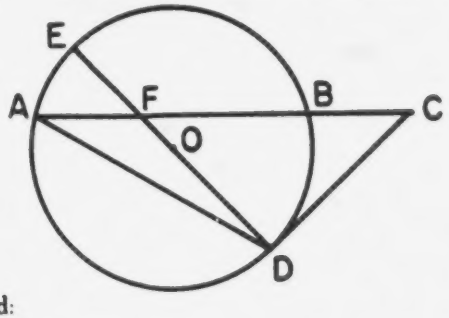
b Graph $\overline{A'B'}$, the image of $\overline{AB}$, after a reflection over the line $y = x$. [2]

c Graph $\overline{A''B''}$, the image of $\overline{A'B'}$, after the transformation $(x,y) \rightarrow (x - 5, y - 5)$. [2]

d Graph $\overline{A'''B'''}$, the image of $\overline{A''B''}$, after a reflection through the origin. [2]

e Write a translation which will map $\overline{A'B'}$ onto $\overline{B'''A'''}$. [2]

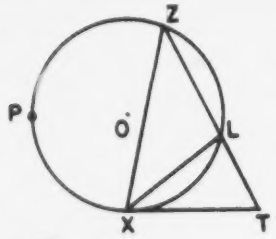
13 In the accompanying diagram, $\overline{CD}$ is tangent to circle O, secant $\overline{ABC}$, chord $\overline{AB} \cong$ chord $\overline{AD}$, $m\widehat{AB} = 150$, and chord $\overline{EOD}$ is a diameter.



Find:

a $m\widehat{AD}$ [2]
b $m\angle BAD$ [2]
c $m\angle ACD$ [2]
d $m\angle EFB$ [2]
e $m\angle EDA$ [2]

14 Given: circle O, tangent $\overline{TX}$, secant $\overline{TLZ}$, chords $\overline{ZX}$ and $\overline{XL}$, $m\widehat{XL}:m\widehat{LZ}:m\widehat{XPZ} = 2:2:5$.



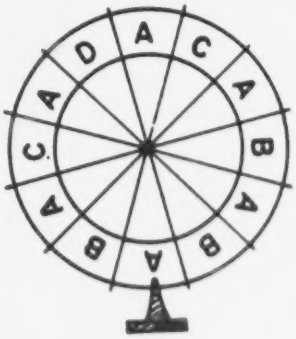
Find: *a* $m\widehat{XL}$ [2]
b $m\angle Z$ [2]
c $m\angle T$ [2]
d $m\angle ZXT$ [2]
e $m\angle XLT$ [2]

15 *a* Sketch and label the graph of $y = 3^x$. [4]

b On the same set of axes, sketch the reflection of the graph of $y = 3^x$ over the line $y = x$. [4]

c Write the equation for the reflected graph in part *b*. [2]

16 The wheel shown in the accompanying diagram is divided into twelve regions of equal area labeled as shown. On any spin of the wheel, each of the regions is equally likely to stop at the arrow.



a Find:
(1) $P(A)$ [1]
(2) $P(B)$ [1]

b Find the probability of:
(1) getting *at least* three *A*'s in four spins of the wheel [4]
(2) getting *no more than* one *B* in three spins of the wheel [4]

17 The numeric key pad on a calculator is arranged as shown in the diagram below. The probability of pressing any key at random is the same for each key.

a Find:
(1) $P(6)$ [1]
(2) $P(\text{even number})$ [1]
(3) $P(\text{odd number})$ [1]

b Find the probability of:
(1) pressing exactly 2 even numbers on three random presses [3]
(2) getting at least 2 even numbers on three random presses [4]

COURSE III

Scope and Content

I. Complex Numbers (recommended time: 25 days)

- A. Further study of the real number system
- B. Extension of algebraic techniques
- C. Introduction of the complex number system
- D. Quadratics with complex roots
- E. Absolute value: equations and inequalities

II. Relations and Functions (recommended time: 25 days)

- A. Definitions and properties
- B. Exponential and logarithmic functions
- C. Common logarithms and computation

III. Circular Functions (recommended time: 60 days)

- A. Further work with the geometry of the circle
- B. Circular functions and their graphs
- C. Trigonometric identities and equations
- D. Applications to area, Law of Sines, Law of Cosines, right triangle trigonometry

IV. Transformation Geometry (recommended time: 20 days)

- A. Review of mapping
  1. Definitions, one-to-one, identity mappings, inverse mappings
  2. Some geometric examples
- B. Line Reflections
  1. Considered as mappings
  2. Properties "preserved" by line reflections
  3. Line symmetry
  4. Coordinates of image of point (x, y) under reflection in special lines of coordinatized plane

C. Translations

1. Considered as mappings
2. Properties "preserved" by translations
3. Translations as composition of line reflections
4. Coordinates of image of point (x, y) under translation

D. Rotations

1. Considered as mappings
2. Properties "preserved" by rotations
3. Rotations as composition of line reflections
4. Coordinates of image of point (x, y) under special rotations
5. Rotational symmetry

E. Isometries

1. Definition
2. Relation to congruence
3. Direct isometries
4. The three-line reflection theorem
5. Dilations as examples of transformations which are not isometries
6. Coordinates of image of point (x, y) under dilation
7. Relation to symmetry

F. Applications to Group Theory

1. The identity transformation
2. Transformations whose composition is the identity
3. The group of translations
4. The group of rotations
5. The groups of isometries; of direct isometries

V. Probability and Statistics (recommended time: 20 days)

A. Review of previous work with probability, combinatorics

B. Probability of exactly r successes in n trials

C. The Binomial Theorem

D. Review of previous work with statistics

E. Measures of dispersion:

1. range, mean absolute deviation
2. variance
3. standard deviation

F. Σ - notation

VI. Optional Topics

A. Polar coordinates

B. Number theory

C. Mathematical induction

D. Sequences and series

E. Matrices

F. Synthetic substitution, remainder theorem, and factor theorem

CO

SEQUENTIAL MATHEMATICS COURSE III TEXT REFERENCES

PART 1

Lesson Number	Amsco Pages	Houghton - Mifflin Pages	Merrill Pages
1	1-6	1-5	2-5
2	6-12	10-12	-
3	13-18	13-14	6-8
4	-	15-20	6-9
5	124-127	21-25	6-9
6	124-127	25-28	6-9
7	127-131	32-35	10-12
8	132-135	32-35	10-12
9	423-431	41-46	136-139
10	26-30, 33-34	46-50	13-15
11	35-39	50-54	16-18
12	171-175	54-59	17-18, 47,49
13	30,34	60-64	-
14	46-51	64-67	24-26
15	51-61	68-71	27-29
16	61-65	71-73	30-32
17	65-71	74-78	33-36
18	641-646	78-81	37-40
19	148-157	85-89	42-45
20	157-165	90-93	46-49
21	175-179	94-97	50-52
22	600-605	97-100	53-56
23	605-615	100-104	57-59
24	615-620	104-107	60-62
25	628-633	107-112	63-67
26	634-640	113-117	68-70
27	222-228	126-129	80-83
28	228-235	121-125	84-87
29	237-240	129-134	88-90
30	234,236	126-129	85-87
31	250-255	138-140	99-102
32	256-262	141-146	95-98
33	-	134-138	91-94

34	182-188	153-158	106-111
35	189-194	158-163	106-108
36	195-200	164-168	112-115
37	206-210	176-180	116-119
38	210-214	181-186	120-123
39	214-218	192-197	130-132
40	73-79	201-206	172-174
41	79-86	201-206	172-174
42	79-86	224-225	184-187, 191,193
43	79-86	224-225	184-187, 191,193
44	81-86	224-230	184-187
45	81-86	224-230	184-187
46	86-92	207-212	188-190
47	86-92	207-212	188-190
48	93-100	230-236	179-183
49	93-100	230-236	179-183
50	100-109	219-224	192-194
51	100-109	219-224	192-194
52	109,112,114	212-218	191-193
53	110-115	224-236	185,186,187,194
54	110-115	224-236	185,186,187,194
55	-	224-236	185,186,187,194
56	-	237-242	175-178
57	-	237-242	175-178
58	-	242-246	195-200
59	-	242-246	195-200
60	301-304	353-358	268-272
61	322-326	287-292	209-210
62	305-308	251-254	204-207
63	344-352	254-256	205-207
64	310-322	257-265	225-227
65	366-373	-	-

TEACHING FUNCTIONS

THE OVERHEAD PROJECTOR

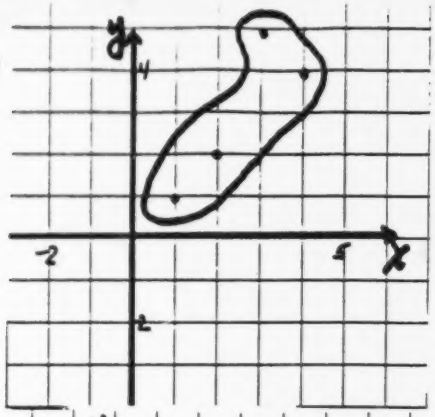
ADVANTAGES:

1. Simplicity of Operation.
2. No change in classroom lighting.
3. A horizontal writing surface.
4. Position of the teacher.
5. Advance preparation of materials.
6. Utilization of dynamic effects.
7. Saving of time during the period.

SOME SUGGESTIONS FOR USE OF THE OVERHEAD PROJECTOR:

1. Can be used by students when they are assigned the explanation of homework problems. This saves the time that would ordinarily be taken by having students write solutions to homework on the chalkboard. Also the projected solutions are more easily viewed than those placed on the chalkboard.
2. Can be used by the teacher by writing the "do-now" questions on an acetate in advance of the class. In this way the teacher can devote time to students in the room as they begin immediately to work on the "do-now".
3. Can be used by the teacher with graphing by having a prepared grid available for immediate use.
4. Movement of geometric models may be done using overlays of transparencies.
5. Can be used by the teacher by writing any verbal problems on transparencies and thus save time from writing them on the board.
6. Can be used by the teacher in copying a model problem from a text and discussing how to use the model problem with the class. This saves having students bring their texts to class.

1)



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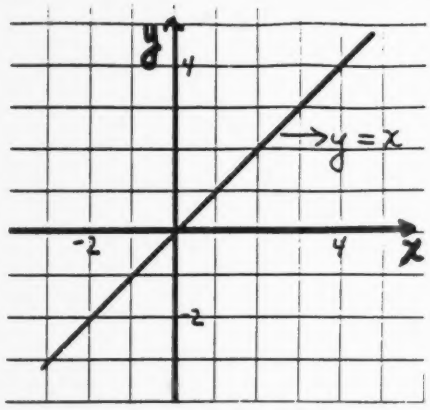

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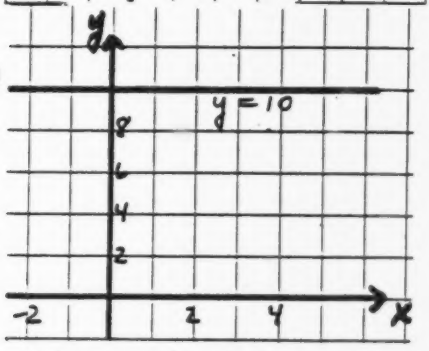
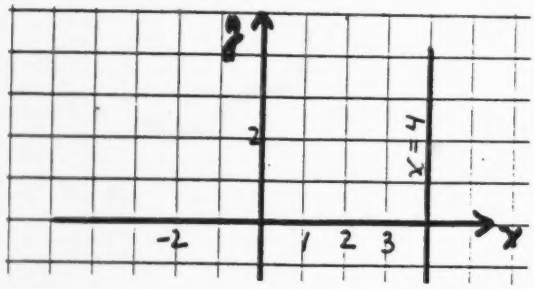


(The graph of a Constant function is a horizontal line)

4)



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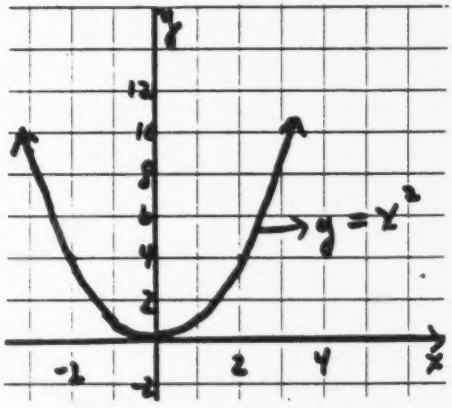

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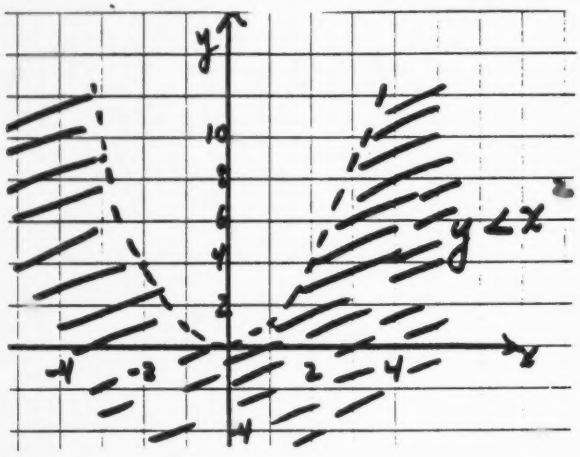

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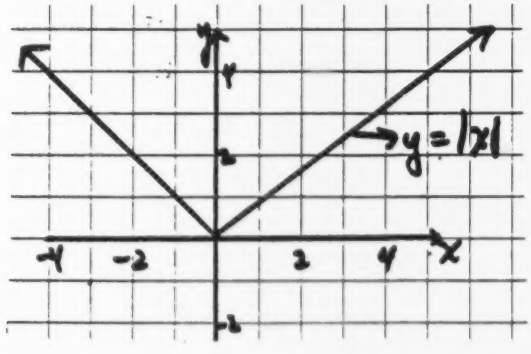


TEACHING INVERSES OF FUNCTIONS

HOW SHOULD WE TEACH COURSE III STUDENTS TO SOLVE PROBLEMS SUCH AS THE FOLLOWING?

1. Given: $F(x) = 4x + 8$. What is the inverse function, $F^{-1}(x)$?

2. Given $y = 2x - 1$. What is the inverse of this function?

3. What is the inverse of $y = x^2 - 5x + 6$?

4. How do we describe and define the function which is the inverse of $y = \cos x$?

5. Write the domain, range and inverse of the function $y = 2^x$.

SAMPLE TEST (FUNCTIONS, INVERSES) Name_____

Show all work

I 1) Discuss the nature of the roots of $x^2 - 3x + 6 = 0$. (5 pts.)

2) Determine the positive value of k in $x^2 + kx + 25 = 0$ that will make the roots equal. (5 pts.)

3) Find S, the sum of the roots and P, the product of the roots of $2x^2 - 7x - 2 = 0$. (4 pts.)

4) One root of $x^2 - 4x + k = 0$ is $2 + 3$.
a) What is the other root? (2 pts.)
b) Find k. (3 pts.)

5) Write a quadratic equation with integral coefficients whose roots are: (5 pts each)
a) $4 + 2$ and $4 - 2$
b) $1 + i$ and $1 - i$

II. For each relation find a) the domain and b) the range (5 pts each)

1. $y = 8x + 3$

2. $y = \frac{1}{x-2}$

3. (2,5), (-3,2), (6,9)

4. $x = y^2$

III. Which of the following are functions? (6 pts.)

1) $y - 2x + 3$ _____
2) $y = x^2$ _____
3) $x^2 + y^2 = 25$ _____
4) $y = 3$ _____
5) $x = y^2$ _____
6) $x = 10$ _____

IV. If $f(x) = 4x^2$, find:

a) f(2)

b) f(-3)

c) f(a)

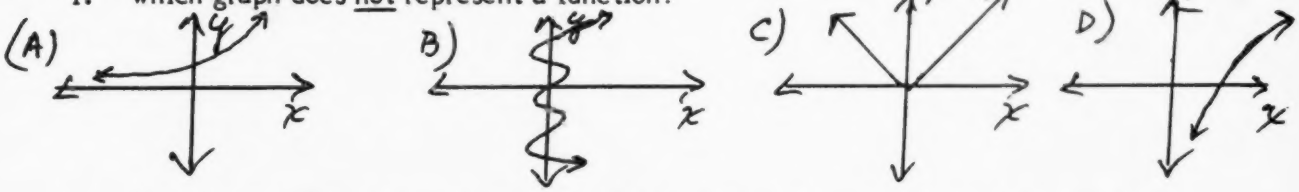
(6 pts)

V. Solve for x and express the roots in the form a + bi (15 pts.)

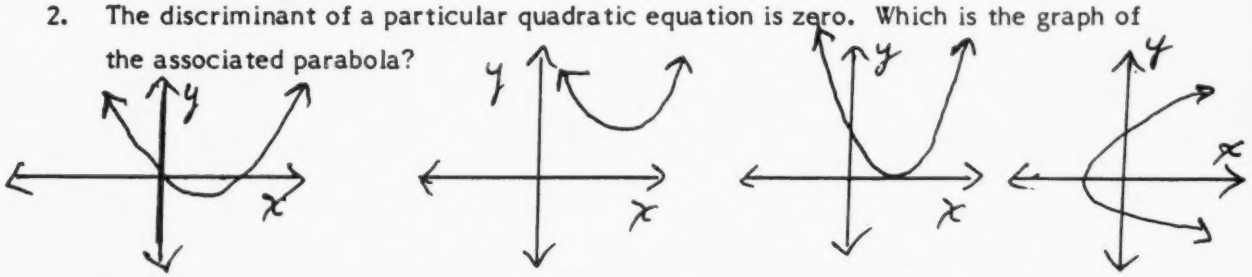
$$x + \frac{5}{x} = 2$$

VI. Multiple Choice. (3 pts each)

1. Which graph does not represent a function?



2. The discriminant of a particular quadratic equation is zero. Which is the graph of the associated parabola?



VII. $f(x) = \frac{x}{2}$ $g(x) = x^2$ $h(x) = x + 2$ (20 pts)

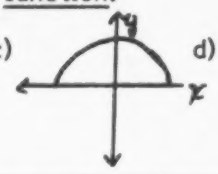
Find: 1. $f \circ g$ 2. $g \circ f$ 3. $g \circ h$

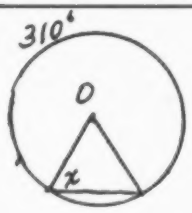
4. $g(h(0))$ 5. $f(g(5))$

SAMPLE TEST (FUNCTIONS, INVERSES) Name_____

		Answers
(1) $f(x) = \frac{1}{x+5} + \frac{x-3}{x-4}$.	What number or numbers are not included in the domain of f(x)?	1
(2) $g(x) = \sqrt{x-2}$.	Find the domain of g.	2
(3) $h(x) = \lvert x+2 \rvert$,	Find the range of h.	3
(4) $f(x) = 4 - 3x^2$.	Find $f(\sqrt{2})$.	4
(5) $g(x) = 2x^2 + \lvert x-1 \rvert + 5x$.	Find g(-3)	5
(6) $f(x) = 5x - 2$. $g(x) = x^2 - 1$.	Find f (g(-2)).	6
(7) $g(x) = 2x + 3$.	Express $g^{-1}(x)$ in terms of x	7

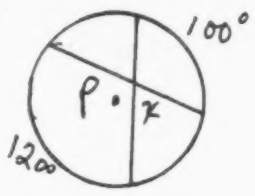
(8) Multiple Choice: Indicate the letter of the choice which is not the graph or not the equation of a function. 8

a) $y = 2x + 1$ b) $y = \lvert x - 1 \rvert$ c)  d)

(9) 

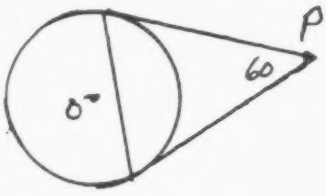
Given Circle 0, Find the value of x

$x =$

(10) 

Given Circle P, Find the value of x

$x =$

(11) 

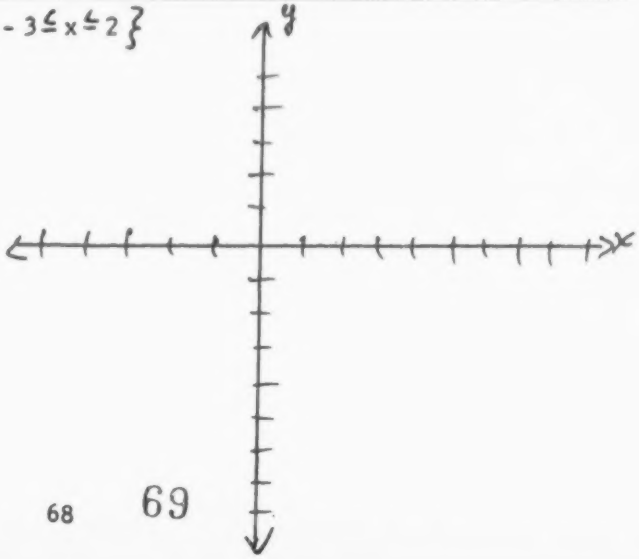
In circle 0, m∠P = 60 Tangents $\overline{PA}$, $\overline{PB}$, Chord $\overline{AB}$. PA = 10. Find the perimeter of △ PAB

12 The domain of a function f, is $\{x: -3 \leq x \leq 2\}$

$$f(x) = \begin{cases} 3 \text{ if } -3 \leq x \leq -1 \\ x \text{ if } -1 \leq x < 1 \\ 2 - x^2 \text{ if } 1 \leq x \leq 3 \end{cases}$$

Complete the following table:

x	f(x)
-3	
-2	
-1	
0	
1	
2	



SAMPLE TEST (FUNCTIONS, INVERSES) Name_____

PART I For multiple choice questions, write the numeral preceding the best answer. For other questions, write your answer in the space provided at the right. (5 points each) If you omit one question, you may still get 100%.

Answers

(1) What is the domain of $f(x) = \sqrt{x - 2}$? 1.

(1) $\{x: x > 0\}$ (2) $\{x: x \geq 2\}$ (3) $\{x: x \leq 2\}$ (4) $\{x: x \geq -2\}$

(2) $f(x) = \frac{x - 2}{x - 5} + \frac{x - 3}{x - 4}$. The set of numbers that must not be in the domain of f is: 2

(1) $\{2, 3, 4, -5\}$ (2) $\{2, 3, 4, 5\}$ (3) $\{0\}$ (4) $\{4, -5\}$

(3) $F(x) = x^2 - 3x$. Find $F(-2)$ 3

(4) $F(x) = x - 4$. For what value of x is $F(x) = 4$? 4

(5) The domain for $f(x) = 2x - 1$ is $\{-3 \leq x \leq 3\}$. What is the greatest value in the range of f? (1) 5 (2) -12 (3) -7 (4) 12 5

(6) What is the range of the function $y = \sqrt{x - 1}$ where $x \geq 1$? 6

(1) $y \geq 1$ (2) $y \geq 0$ (3) $y \leq 0$ (4) all real numbers

(7) What is the inverse of the function $\{(-1, 2), (0, 5) (3, -2)\}$? 7

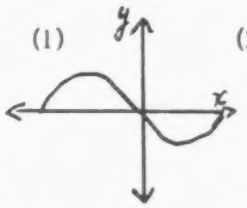
(1) $\{(1, -2), (0, -5), (-3, 2)\}$ (2) $\{(-1, -2), (0, -5), (3, 2)\}$

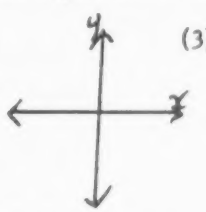
(3) $\{(2, -1), (5, 0), (-2, 3)\}$ (4) $\{(-2, 1), (-5, 0), (-2, -3)\}$

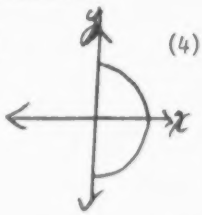
(8) What is the inverse of the function $y = 2x + 1$? 8

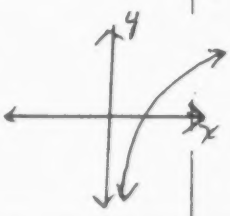
(1) $y = \frac{1}{2}x$ (2) $y = 2x - 1$ (3) $y = \frac{x - 1}{2}$ (4) $y = \frac{x + 1}{2}$

(9) Which graph does not represent a function? 9

(1) 

(2) 

(3) 

(4) 

(10) Express $i^3 + i(2 - i)$ in a + bi form. 10

(11) Express $\frac{\frac{1}{2}}{\frac{1}{x} - 1}$ in simplest form. 11

(12) Solve: $|2x - 1| = 3$ 12

70

(13) Solve: $\frac{1}{15} + \frac{1}{10} = \frac{1}{x}$	13
(14) If the graph of $y = 2x^2 + 4x + c$ does not intersect the x-axis, then c may be equal to (1) 1 (2) 2 (3) -2 (4) 5	14
(15) Factor completely: $4x^3 - x$	15

Part II Show All Work

(16) Solve: $x^2 - 2x + 5 = 0$. Answer in a + bi form. 10 points

(17) The domain of a functon g is $\{x: -1 \leq x \leq 3\}$

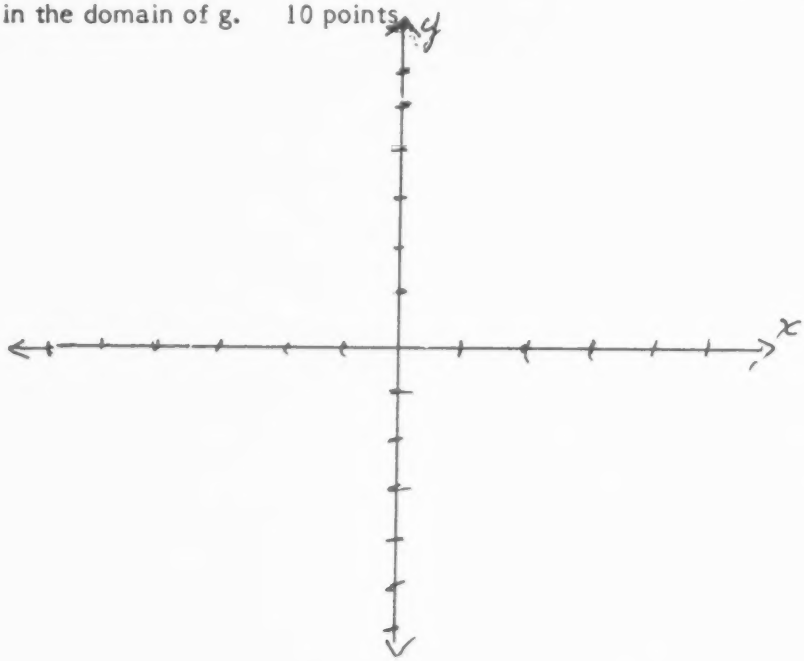
$$g(x) = \begin{cases} 2x + 1 \text{ if } -1 \leq x \leq 1 \\ 3 \text{ if } 1 \leq x \leq 3 \end{cases}$$

(a) Fill in the table below. 10 points

(b) At the right, graph g(x) for all values of x in the domain of g. 10 points

DRAW The Graph of f(x)

x	g(x)
-1	
0	
1	
2	
3	



SAMPLE TEST { QUADRATIC EQUATIONS / COMPLEX NUMBERS } Name_____

PART I Express in a + bi form

1. Subtract 1 + 2i from the sum of 2 - 3i and -2 + 5i
2. The additive inverse of -1 + 2i
3. $\sqrt{-81} + 2\sqrt{-25}$
4. The product of 3 + 2i and 3 - 2i
5. $\frac{7}{3 - 4i}$
6. $(1 + i)^2$
7. $i^{100} + i^{101}$
8. The roots of $x^2 + 2x = -10$.
9. The conjugate of 4 - 7i

PART II Write a quadratic equation in standard form with integral coefficients whose roots are:

10) $\frac{1}{2}$ and -2

11) 2 + i and 2 - i.

PART III

12. In how many points does the graph of $y = x^2 - 3x - 4$ intersect the x-axis?
13. (Multiple Choice) The graph of a parabola ($y = ax^2 + bx + c$) and a horizontal line ($y = k$) are drawn on the same set of axes. They do not intersect. The roots of $ax^2 + bx + c =$ are (A) real and $=$, (B) rational but $\neq$, (C) imaginary, (D) can't tell from given information.
14. An equation of the axis of symmetry of $y = 2x^2 - 6x + 4$ is

    (A) $x = 6$ (B) $x = \frac{-3}{2}$ (C) $x = \frac{3}{2}$ (D) $y = 4$
15. Solve: $\frac{x}{2} - \frac{x}{6} = 5$
16. Solve: $\sqrt{x} + 1 = 0$

TEACHING

GEOMETRIC TRANSFORMATIONS

MOTIVATIONAL TRANSFORMATION PROBLEMS

1. THE COWBOY

A TIRED COWBOY (POINT A) IS RETURNING TO HIS CABIN (POINT C). HE WISHES TO STOP AT THE BANK OF STREAM s TO TAKE WATER FOR HIMSELF AND HIS HORSE. NATURALLY, HE WISHES TO TRAVEL THE SHORTEST POSSIBLE ROUTE FROM A TO C.

PROBLEM: HOW SHOULD HE LOCATE POINT B ON THE BANK OF THE STREAM?

2. MINNESOTA FATS

MINNESOTA FATS, WHO EVIDENTLY KNOWS SOME GEOMETRY, WRITES:

> TO MAKE A BANK SHOT IN POCKET BILLIARDS WITH OBJECT BALL O TO BE BANKED INTO POCKET A AFTER CAROMING OFF CUSHION BC:
>
> i. VISUALIZE $\overline{OC} \perp \overleftrightarrow{BC}$ ii. VISUALIZE $\overline{OB}$ iii. VISUALIZE $\overline{CA}$
> iv. FROM THE INTERSECTION OF $\overline{OB}$ AND $\overline{CA}$, VISUALIZE $\overline{DX}$, THE SEGMENT PERPENDICULAR TO $\overline{BC}$
> v. POINT X, THE FOOT OF THE PERPENDICULAR, IS THE CRITICAL SPOT AT WHICH TO AIM THE OBJECT BALL TO SUCCEED.
> (OF COURSE, WE ASSUME SUFFICIENT FORCE AND NO ENGLISH.)

PROBLEM: USE TRANSFORMATION GEOMETRY TO SIMPLIFY THE EXPLANATION.

3. ROTATION OF A LINE SEGMENT 90° CLOCKWISE ABOUT A POINT

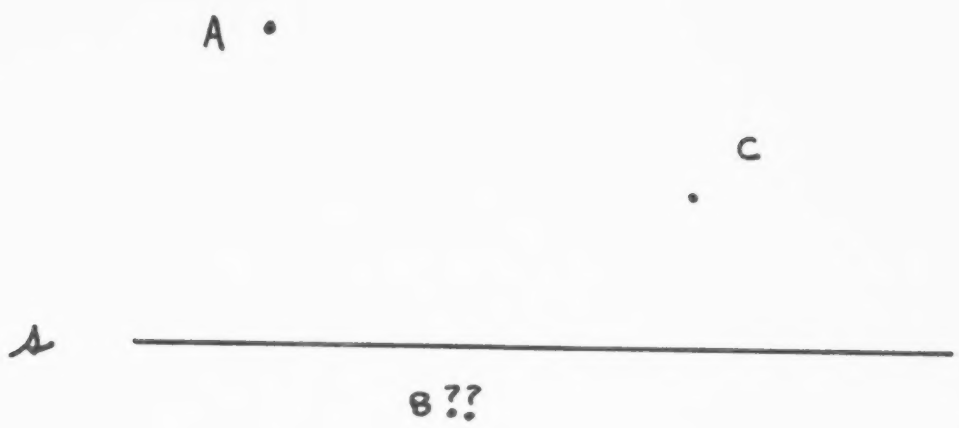
PROBLEM: GIVEN ANY LINE SEGMENT $\overline{AB}$, ROTATE $\overline{AB}$ 90° CLOCKWISE WITH P THE CENTER OF ROTATION.

i. DRAW $\overrightarrow{AP}$
ii. CONSTRUCT 90° ANGLE WITH VERTEX P AND ONE SIDE $\overrightarrow{AP}$
iii. ON TERMINAL SIDE OF THIS ANGLE LOCATE A′ SO THAT $\overline{A'P} \cong \overline{AP}$.
iv. REPEAT THE PROCESS TO LOCATE B′, THE IMAGE OF B.

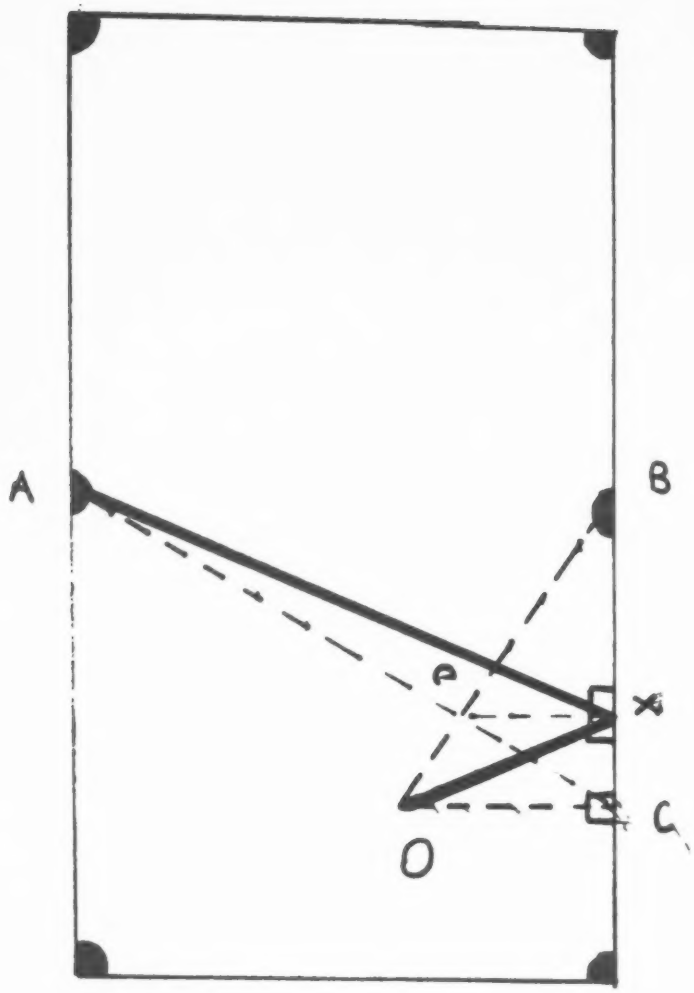
FOLLOW UP PROBLEM: SHOW HOW A 180° ROTATION ABOUT THE ORIGIN IS USEFUL IN EXPLAINING TO STUDENTS THE CONCEPT OF REFERENCE ANGLES FOR THIRD QUADRANT ANGLES.

1.

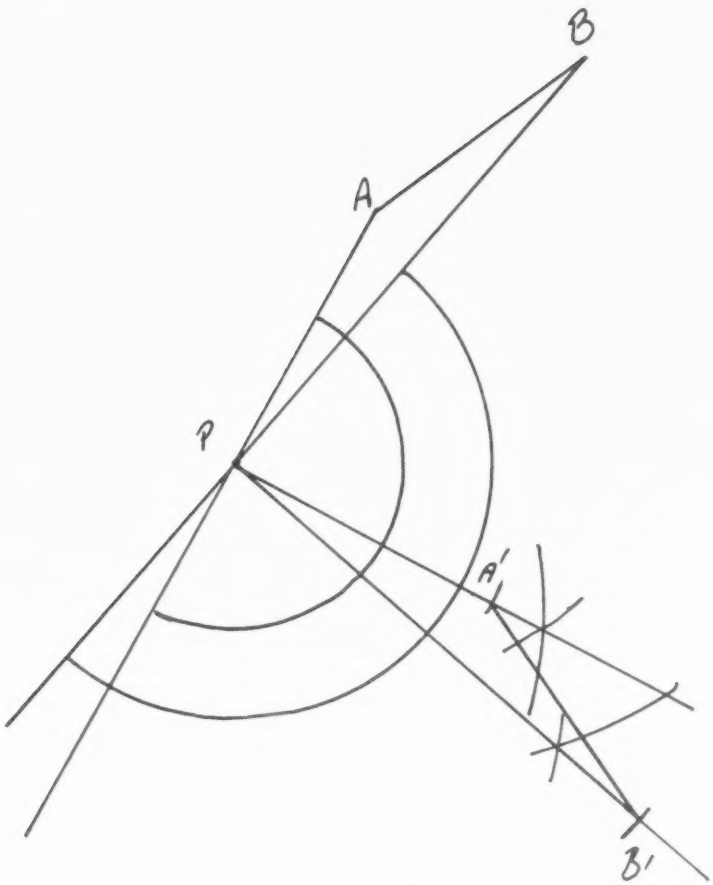
THE COWBOY



2. MINNESOTA FATS

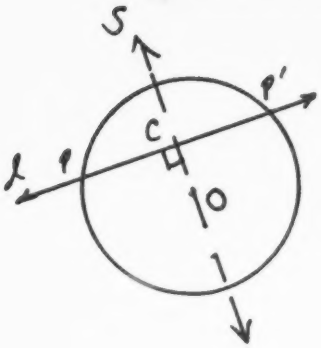


3. ROTATION 90° CLOCKWISE



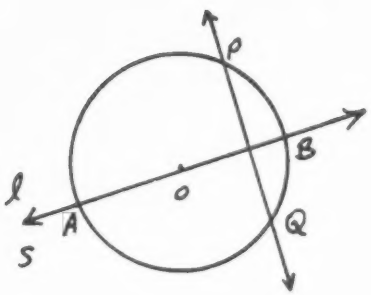
USING TRANSFORMATIONS TO DEMONSTRATE PROPERTIES OF CIRCLES

1. If a line $\overleftrightarrow{l}$ intersects a circle at a point, then the only other point at which the line intersects the circle is the point's image under $R_{\overleftrightarrow{s}}$, where $\overleftrightarrow{s}$ is the line which contains the center and which is perpendicular to $\overleftrightarrow{l}$.



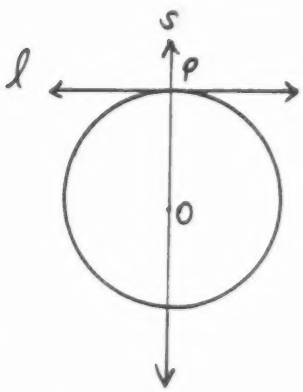
Suppose $\overleftrightarrow{l}$ intersects circle O at P.
There exists a line s that contains O and which is perpendicular to l, say at C.
Circle O and line l are symmetric with respect to line s.
Let $R_s(P) = P'$.
P' is the only other point at which line l intersects circle O.

2. If a line containing the center of a circle is perpendicular to a chord of the circle, then it bisects the chord and its arcs.



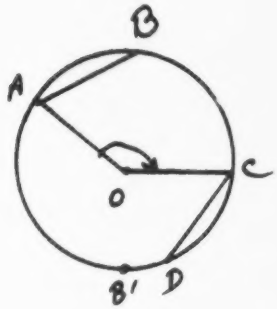
Given: $\overleftrightarrow{PC} \perp \overleftrightarrow{s}$, $O \in \overleftrightarrow{s}$, $P \in \odot O$
Prove: $\overline{PC} \cong \overline{QC}$
Let A', B', C', and P' be the images under R_s of A, B, C, and P, respectively.
Since $\overleftrightarrow{PC} \perp \overleftrightarrow{s}$, $P' \in \overleftrightarrow{PC}$.
But P' is the only other point at which $\overleftrightarrow{PC}$ intersects circle O.
So P' = Q.
Also, A' = A, B' = B, and C' = C.
So $\overline{PC} \cong \overline{QC}$, $\overline{PA} \cong \overline{QA}$, and $\overline{PB} \cong \overline{QB}$.

3. If a line is perpendicular to a line containing the center of a circle at a point on the circle, then it is a tangent.



Given: $\overleftrightarrow{l}$ intersects circle O at P, $\overleftrightarrow{l} \perp \overline{OP}$.
Prove: $\overleftrightarrow{l}$ is tangent to circle O.
Let P' be the image of P under $R_{\overleftrightarrow{l}}$.
P' is the only other point at which $\overleftrightarrow{l}$ intersects circle O.
But P = P'.
So P is the only point at which $\overleftrightarrow{l}$ intersects circle O.
$\overleftrightarrow{l}$ is tangent to circle O at P.

4. In the same or congruent circles, if two arcs are congruent, then their chords are congruent.



Given: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$

Prove: $\overline{AB} \cong \overline{CD}$

Let A' and B' be the images under rot_O, $m\measuredangle AOC$ of A and B, respectively.

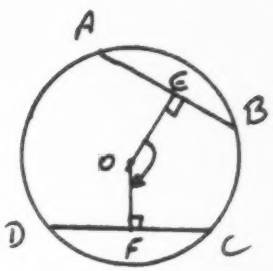
Since A' = C and $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$, $\overset{\frown}{AB} \cong \overset{\frown}{CB'}$.

Also, $\overset{\frown}{AB} \cong \overset{\frown}{CD}$, so $\overset{\frown}{CB'} \cong \overset{\frown}{CD}$.

Therefore, B' = D.

Since $\overline{AB} \cong \overline{A'B'}$, $\overline{AB} \cong \overline{CD}$.

5. If two chords of a circle are equidistant from the center, then they are congruent.



Given: $\overline{OE} \cong \overline{OF}$, $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$.

Prove: $\overline{AB} \cong \overline{CD}$.

Let A', B', C', D', E', F', and O' be the images under rot_O, $m\measuredangle EOF$ of A, B, C, D, E, F, and O, respectively.

E' = F, O' = O.

Since $E \in \overline{AB}$, $E' \in \overline{A'B'}$, that is, $F \in \overline{A'B'}$.

Since $\overline{OE} \perp \overline{AB}$, $\overline{O'E'} \perp \overline{A'B'}$, that is, $\overline{OF} \perp \overline{A'B'}$.

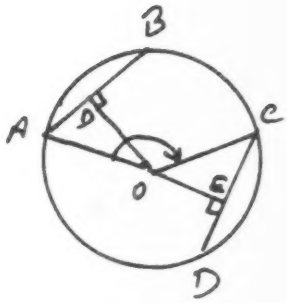
But $\overline{OF} \perp \overline{CD}$

So $\overleftrightarrow{A'B'} = \overleftrightarrow{CD}$.

Since A' is on circle O and B' is on circle O, A' = C and B' = D.

So $\overline{AB} \cong \overline{A'B'}$, that is, $\overline{AB} \cong \overline{CD}$.

6. If two chords of a circle are congruent, then they are equidistant from the center.



Given: AB = CD, $\overline{OD} \perp \overline{AB}$, $\overline{OE} \perp \overline{CD}$.

Prove: OD = OE.

Let A', B', C', and D' be the images under rot_O, $m\measuredangle AOC$ of A, B, C, and D, respectively.

$\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

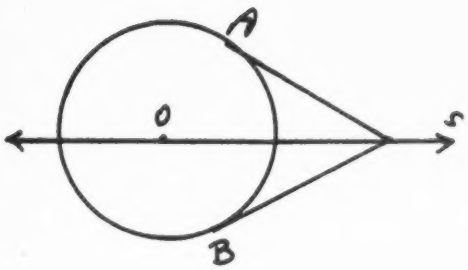
As in (4), A' = C and B' = D.

From (2), D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{CD}$.

Since D' is the midpoint of $\overline{A'B'}$, D' is the midpoint of $\overline{CD}$.

So D' = E, and $\overline{OD} \cong \overline{OE}$.

7. If two tangent segments share an endpoint in the exterior of a circle, then they are congruent.



Given: $\overline{PA}$ tangent to circle O at A.
$\overline{PB}$ tangent to circle O at B.

Prove: $\overline{PA} = \overline{PB}$.

Let P' and A' be the images under R_s of P and A, respectively.

If $\overline{PA}$ is tangent to circle O at A, then $\overline{P'A'}$ is tangent to circle O at A'.

So A' = B.

Since $\overline{PA} \cong \overline{P'A'}$ and $P' \neq P$, $\overline{PA} \cong \overline{PB}$.

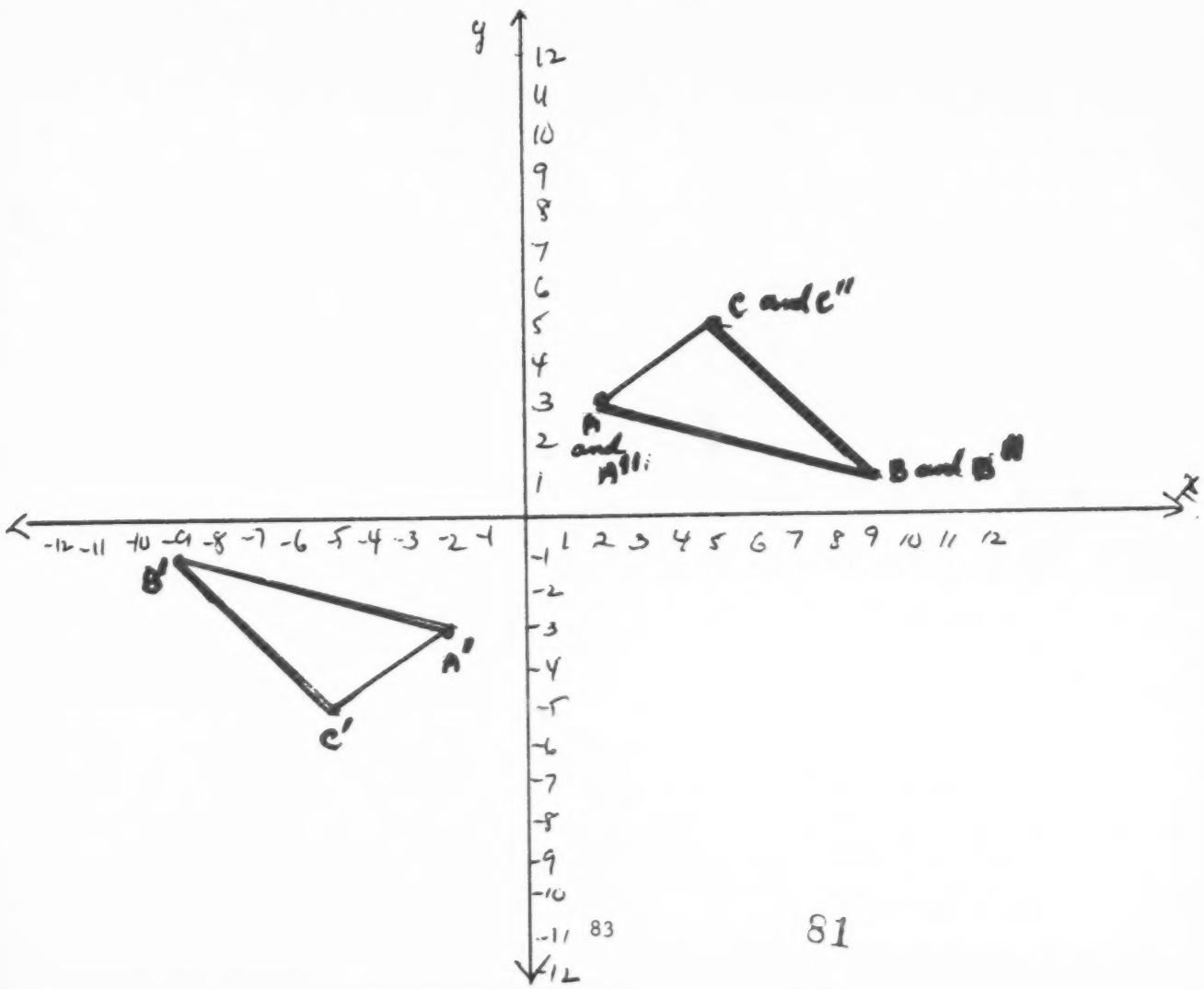
TRANSFORMATION PROBLEMS - COORDINATE GEOMETRY

△ABC has coordinates A (2,3), B (9,1) and C(5,5).

a) (1) On graph paper, draw and label △ABC.
(2) Find the coordinates of A'B'C', the image of △ABC after a reflection through the origin.
(3) Find the coordinates of △A"B"C", the image of △A'B'C' after $r_{y\text{-axix}}$ or $r^{\triangle}_{x\text{-axis}}$(A'B'C')

b) A series of transformations tht reflect point P through the origin, then over the x-axis, then over the y-axis, is equivalent to a rotation about the origin of

(1) 90° (2) 180° (3) 270° (4) 260°



TRANSFORMATION PROBLEMS - COORDINATE GEOMETRY

The vertices of $\triangle ACE$ are A(0.2). C(5,4) and E(6,2)

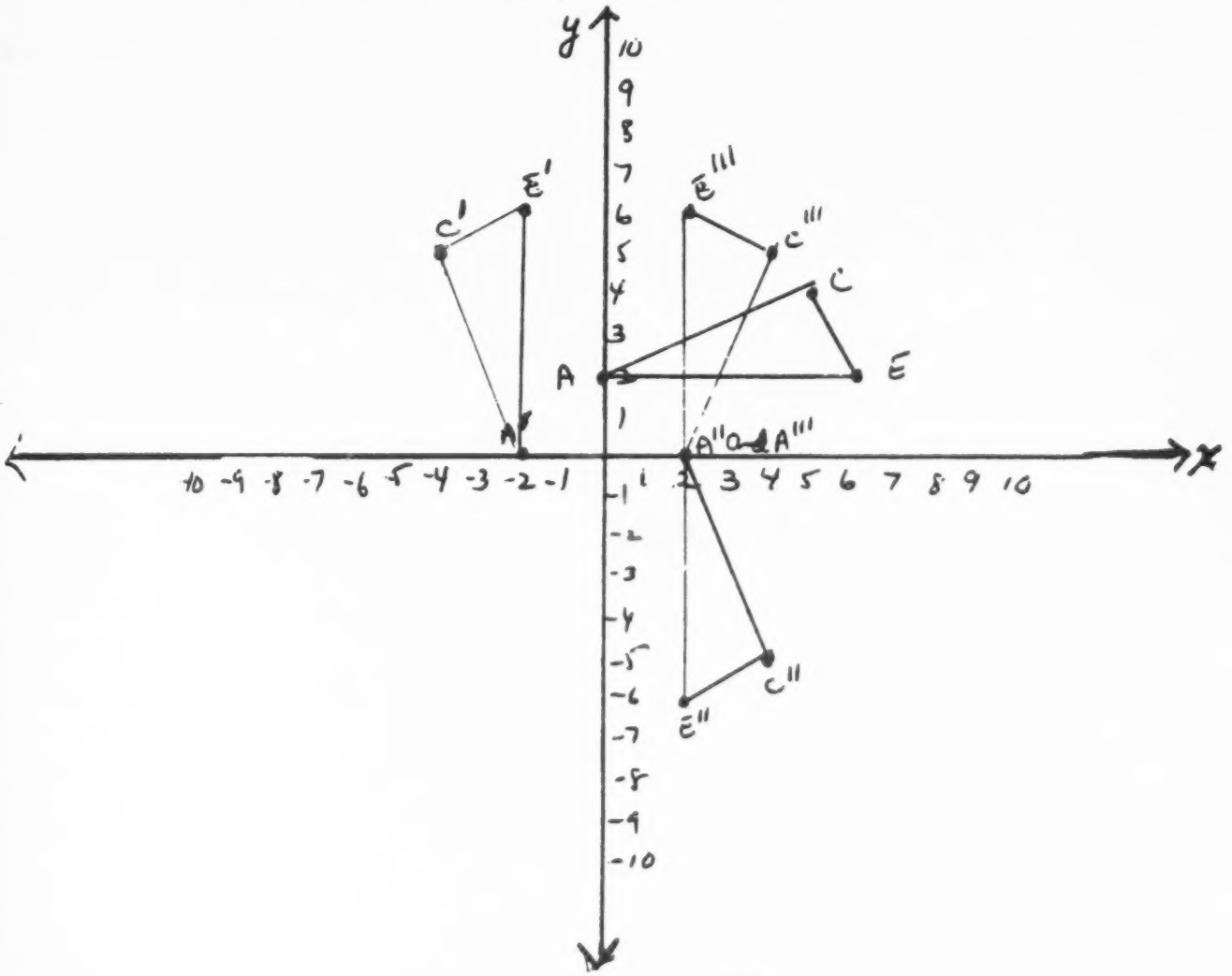
a. Graph $\triangle ACE$ and $\triangle A^1C^1E^1$ where R_{90^0} ($\triangle ACE$) = $\triangle A^1C^1E^1$

b. Graph $\triangle A^{11}C^{11}E^{11}$ such that R_{180^0} ($\triangle A^1C^1E^1$) = $\triangle A^{11}C^{11}E^{11}$

c. Graph $\triangle A^{111}C^{111}E^{111}$ such that $r_{x\text{-axis}}$ ($\triangle A^{11}C^{11}E^{11}$) = $\triangle A^{11}C^{111}E^{111}$

d. The composition $r_{x\text{-aixs}} \circ R_{180^0} \circ R_{90^0}$ is equivalent to the single transformation:

(1) $r_y = x$ (2) $r_y = -x$ (3) r_y- axis (4) R_{270^0}



SAMPLE TEST (TRANSFORMATIONS) Name_____

I. No partial credit. 5 points each

Answers

1. What is the circumference of a circle whose diameter is 10? 1) _____

2. Find the radius of a circle whose area is 36π. 2) _____

3. Which letter has no line symmetry?

a) A b) N c) C d) E 3) _____

4. What kind of symmetry does the word TOOT have?

a) vertical line, only c) both vertical and horizontal
b horizontal line, only d point and vertical line 4) _____

5. Which transformation does <u>not</u> preserve area?

a) rotation b) dilation c) translation d) reflection 5) _____

6. What are the coordinates of the image of the point (-3, 4) after a reflection over the line y = -1? 6) _____

II. - Show work. Partial credit allowed. 5 points each

1. In a circle, an arc of 120° has length 8π. Find the radius of the circle.

2. Find the area of the sector of a circle if the central angle = 30° and the radius = 6.

III. Consider the point (4, -5) 3 points each

Find its image under: 1. $T_{(4, 6)}$ _____ 2. $T_{(-2, 7)}$ _____ 3. R_{90° _____

4. R_{270° _____ 5. $r_{y=x}$ _____ 6. $r_{x\text{-axis}}$ _____

83

IV. 1) Suppose A(x, y) $\rightarrow$ A' under $T_{(2, 5)}$ and A' $\rightarrow$ A" under $T_{(4, -4)}$.
What single translation would map A $\rightarrow$ A"? _____ (4 pts.)

---

2) Suppose Δ PQR, P(1, 1) Q(5, 1) R(4, 3), is reflected through the origin to Δ P'Q'R' and then Δ P'Q'R' undergoes a dilation of 3 to form Δ P"Q"R". What are the coordinates of Δ P"Q"R"? (4 pts.)

---

3. Under a certain translation, the image of P (6, 2) is P' (5, 0). Find the image of R(7, -3) under the same translation. (4 pts.)

V. Given Δ ABC, A (2, 5) B(4, -3) C(-2, 1) (15 credits)

a) Graph Δ ABC

b) Graph and state the coordinates of Δ A'B'C, the image of ΔABC after a reflection on the x-axis.

c) Graph and state the coordinates of Δ A"B"C", the image of Δ A'B'C', after a reflection on the line x=2.

VI. Given Δ ABC A(-2, 3) B(-6, 7) C(-3, 8) (15 credits)

a) Graph Δ ABC

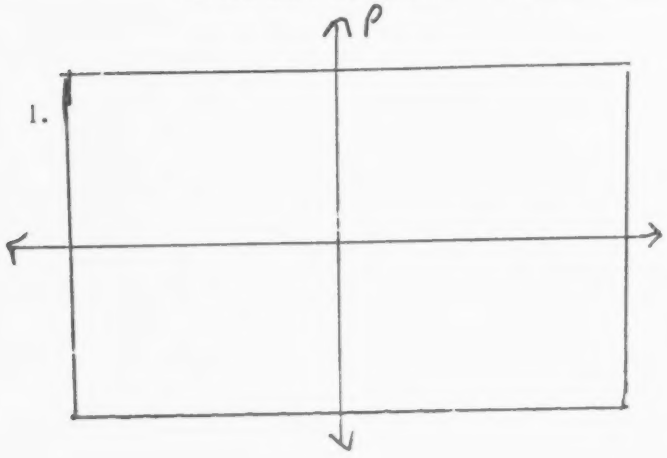
b) Graph and state the coordinates of Δ A'B'C', the image of ABC, after a reflection over the y-axis.

c) Graph and state the coordinates of Δ A"B"C", the image of Δ A'B'C, after a reflection over the line y=x.

d) What single transformation would map ΔABC into Δ A"B"C"?

SAMPLE PROBLEMS - TRANSFORMATIONAL GEOMETRY

1.

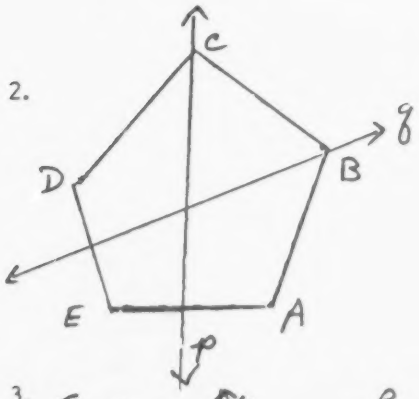


ABCD is a rectangle. P and Q are symmetry lines

Find: $r_p \circ r_q$ (A)

$r_q \circ r_p \circ r_q$ (C)

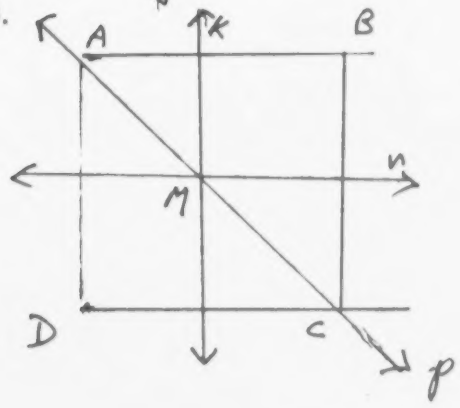
2.



ABCDE is a regular pentagon. P and Q are symmetry lines.

Find $r_q \circ r_p$ (B)

3.



ABCD is a square.

K, N and P are symmetry lines.

Find: a) $r_k \circ R_m$ (B)

b) $R_m \circ r_n \circ r_k$ (A)

c) $R_m \circ R_{m,900}$ (C)

ROTATION AND DILATIONS

1. If point (1,-5) is rotated 90° counter-clockwidse through the origin, what are the coordinates of its image point, P'?

2. $Rot_{O}, 90^{o}$ (x, y) = ? What are the new coordinate of (x, y)?

3. What is the effect of applying a rotation of $\frac{1}{4}$ about the origin to a point A (x, y)?

Find two ordered pairs of <u>integers</u> which satisfy:

4. $41x - 29y = 700$

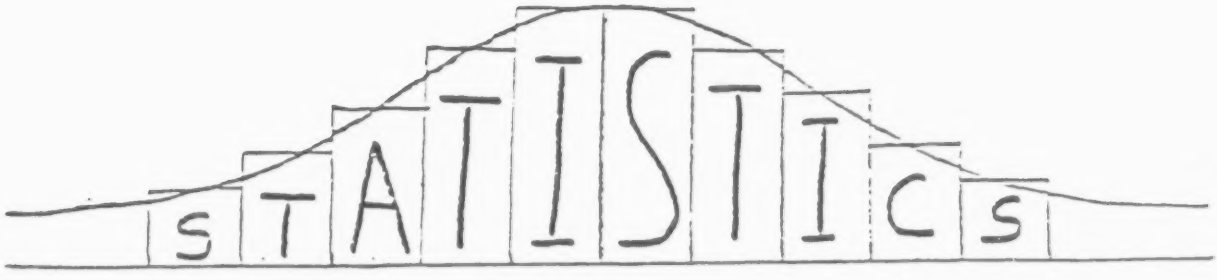
5. $41\frac{x}{2} - 29\frac{y}{2} = 700$

TEACHING PROBABILITY

AND

STATISTICS

AND



Sequential Math

Courses I, II, & III

COURSE I Probability, Permutations, and Statistics (approximate time: 30 days)

A. Discussion of term, probability

B. The counting principle

C. Elementary combinatorics

1. permutations
2. urn problems
   a. with replacement
   b. without replacement

D. Investigation of events whose probabilities must be determined empirically - idea of stabilization of relative frequency

E. Introduction to statistics
1. discussion of term, statistics
2. need and justification for statistics
3. sampling techniques
4. graphical representation and interpretation - histogram

F. Some measures of central tendency
1. mean, median, mode
2. quartiles
3. mathematical and nonmathematical examples

COURSE II Probability, permutations, and Combinations (recommended time: 10-15 days)

A. Review of material on permutations from Course I

B. Permutations of n things taken n at a time with repetition

C. Combinations of n things taken n at a time, r at a time
Notation: ${}_nC_r$, $C(n,r)$, $\binom{n}{r}$; that ${}_nC_r = {}_nC_{(n-r)}$

D. Probability involving combinations

COURSE III Probability and Statistics (recommended time: 20 days)

A. Review of previous work with probability, combinatorics

B. Probability of exactly r successes in n trails

C. The Binomial Theorem

D. Review of previous work with statistics

E. Measures of dispersion:
1. range, mean absolute deviation
2. variance
3. standard deviation

F. Σ- notation

PROBABILITY AND STATISTICS I

A. Foundations of probability
   1. Empirical probability
   2. Concepts and definition
   3. Sample spaces and events

B. Counting Principles

C. Permutations and Application
   1. n factorial
   2. ${}_nP_r$ from an arithmetic viewpoint
   3. Problems with/without replacement

D. Statistics
   1. Sampling techniques
   2. Measures of central tendency (mean, median, mode)
   3. Frequency distribution
      a. Histogram
      b. Cumulative frequency histogram
      c. Quartiles and percentiles

Development

A. Foundations of Probability

The concept of probaility is best introduced on an intuitive basis. Students are aware that:

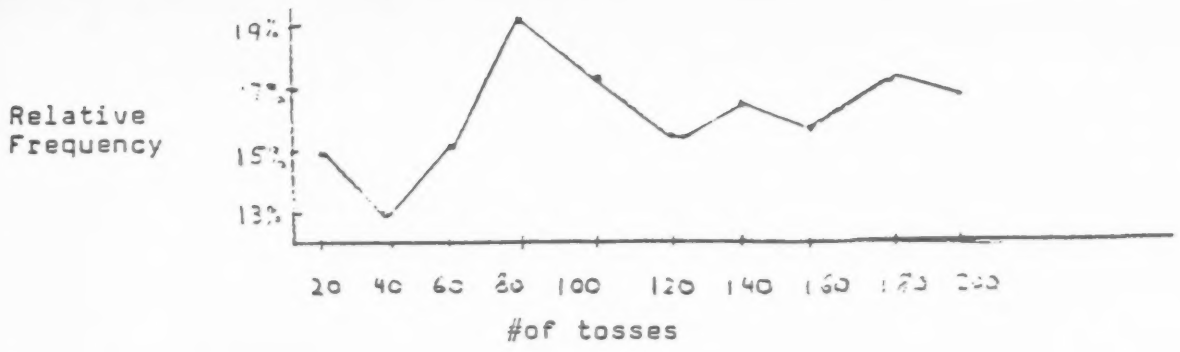
in tossing a coin, heads will appear 1/2 of the time;
in tossing a die, the "4" will appear 1/6 of the time;
in selecting a number from 1 through 10, 1/2 will be even.

Is there any way to prove or demonstrate some of these intuitive feelings' Experiments involving the entire class should follow.

Example: Using small groups or individual, toss fair dice. Record the number of times a "4" appears. Complete a table at the board to show cumulative results. Typical results might look like this:

INDIVIDUAL		CUMULATIVE		RELATIVE FREQUENCY
#4's	#tosses	#4's	#tosses	
3	20	3	20	3/20 = 15.0%
2	20	5	40	5/20 = 12.5%
4	20	9	60	9/60 =15.0%
6	20	15	80	15/80 = 18.8%
2	20	17	100	17/100= 17.0%
1	20	18	120	18/120= 15.0%
5	20	23	140	23/140= 16.4%
3	20	26	160	26/160= 16.3%
4	20	30	180	30/180= 16.7%
3	20	33	200	33/200= 16.5%

(rounded to tenths of a Percent)

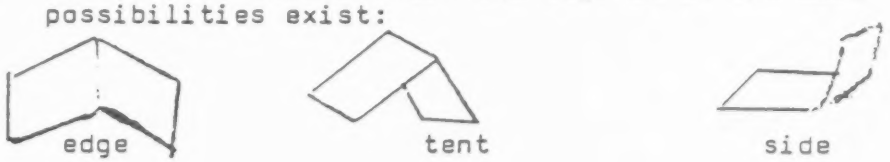


Single experiments can vary greatly. However, the more times the experiment is performed the less the discrepency. This can be seen graphically by the "leveling out" of the line. It can be seen numerically by taking the differences in the relative frequencies. Both cases snow that the differences qpproach zero and the relative frequency converges to a number.

After sufficient experimentation, probability can be thought of as the most accurate sicentific guess of the relative frequency of the event.

After performing experiments where the students can anticipate the answers in advance, the concept should be extended to those kinds of experiments where the students cannot accurately preedict results.

Example: Fold an IBM card in half and toss it in the air. These possibilities exist:



While students might guess 1/3 as the probability for landing on an edge, this will not be demonstrated by the experiment. Enough tosses should be done(at least 20 per student) to arrive at a graph of relative frequency that obviously converges.

After many experiments like these (where the probability can't be determined theoretically) a discussion of what probability is and when it applies should be done. Terms such as sample space, outcome set, and event can be related to the previous experiments. The concept of probability as a ratio will evolve.

B. The Counting Principle

The counting principle can be introduced with a discussion of how many ways an event can occur.

Example: There are 10 doors to the school. How many choices? There are 4 stairways. How many possible ways can you enter the building and go upstairs?

If event M occurs in m ways, and if event N occurs in n ways, then together, events M and N occur in m x n ways.

While the teacher can demonstrate the counting principle with the use of coins, cards, and dice it is advisable to seek out illustrations from the real world as much as possible.

D. Statistics

Whereas probability predicts the chances of events, statistics can be described as the study of data collected from actual events. Students are familiar with statistics:

Example: A batting average of .280 means that this batter would get 280 hits if he were at bat 1000 times. The average might actually have come from 14 actual hits for 50 times at bat.

Since statistics is based on the collection of data, time should be devoted to a discussion of sampling techniques.

Example: If you want to draw conclusions about the height of 9th graders, would the results be the same if the sample were taken in the locker room during a basketball practice as if it were taken in a single homeroom?
How large should the sample be?

Students should be encouraged to bring in samples of the misuse of statistics due to bias in collection of the sample information.

The calculations of the measures of central tendency (mean, mode, median) can be applied to the data collected. When is one more appropriate than another?

Example: If a student has six test scores with grades of B or B+ and one test score of F, which measure of central tendency would she prefer for determining a report card grade?

Once data is collected, the need arises for methods which will help to organize and interpret data.

Example: 150 students took the 9th Year Regents in Mathematics. How could the scores be presented in a meaningful way? The students should see the need to order the scores and set up a frequency distribution chart. From there would develop the histogram and cumulative frequency histogram.

Students may have heard the terms quartiles and percentile use in connection with standardized tests. Calculation of these may be introduced at this time.

Since probability and statistics along with their use and misuse, are such an integral part of our modern world every effort should be made to bring in examples from advertising and the media.

PERMUTATIONS, COMBINATIONS, AND PROBABILITY II

A. Permutations
   1. Formalized notation: ${}_nP_n$, ${}_nP_r$
   2. Applications with and without repetition

B. Combinations
   1. Definition
   2. Notations: ${}_nC_n$, ${}_nC_r$, ${}_nC_{(n-r)}$
   3. Applications

C. Probability - as related to permuations and combinations

In Course I, students have developed sample spaces and tree diagrams, and studied permuations using the counting principle and factorial notation. They calculated measures of central tendency and organized and interpreted data using frequency distributions, historgrams, and percentiles.

In Course II the study of permutations is formalized and extended to include the permutations of things not all different. Combinations, including formalized notation and applications, are studied in conjunction with permuatations and combinations to probability.

Development

A. Permuations

1. Extension of fundamental counting principle
   Emphasis should be placed on "thinking through" each problem. Special parameters must be considered.

   The development of ${}_nP_r$ is a major objective, along with the special case of ${}_nP_n$. This leads into factorial work. The study of permutations of things not all different gives the student practice in manipulating factorials. The formuala $\frac{n!}{a!b!\ldots}$ is used to solve this type of problem

   Example: Calculate the number of distinct permuatation of the letters in the word "TENNESSEE".
   i) n! represents the number of permuatations without regard to duplication.
   ii) Duplicate arrangements are formed when letters that repeat are rearranged.
   iii) Dividing the total number of permutations by the no. of duplicate arrangements produces the total number of distinct arrangements.

   Therefore, $\frac{9!}{4!2!2!}$ represents the number of permutations of the letters in the word "TENNESSEE"

B. Combinations
Emphasis should be placed on the contrast of combinations and permutation:
The notations ${}_nC_r$, $\binom{n}{r}$, C_r^n, and C(n,r) could be noted and the students may see any of them in future math applications. Throughout level II, ${}_nC_r$ is used to represent " the combination of n things, r at a time."

i) ${}_nC_r = \frac{{}_nP_r}{r!}$ ii) ${}_nC_r = \frac{n!}{(n-r)!(r)!}$

i and ii are equivalent and you may use the one you and/or your students are most comfortable with. 0!=1 is a convention needed to develop ${}_nC_n$, ${}_nC_1$, and ${}_nC_0$.

C. Probability
The teacher may find it helpful to employ some visual aids in this unit. Some of the basic materials include: dice, cards, markers of equal size and various colors, blank cubes for making "special" dice, and solids for making "dice" with other than six faces.

At level II, much of the probability involves preparation work with permuations and combinations.

Example: How many different ways can eight boys arrange themselves in a line?

Example: How many different ways can eight boys arrange themselves in a line if two of them are brothers and have to be next to each other?

Example: If eight boys, two of who are brothers, were to line up at random , what is the probability that the two brothers would be next to each other?

PROBABILITY AND STATISTICS III

A. Probability
   1. Review of previous work with permutations, combinations, and probabili
   2. Extension of techniques
      a. Simple "condition " probability
      b. Combinatorics applied to probability
      c. Probability of r successes in n trials of a binomial experiment (exactly, at least, less than, etc.)

B. The Binomial Expansion
   1. Binomial theorem
   2. Relation to combinations
   3. Nth term of an expansion

C. Statistics
   1. Measures of Central Tendency
      a. Review of previous work
      b. Sigma notation
      c. Extension to include grouped data
   2. Measures of Dispersion
      a. Range
      b. Mean absolute deviation
      c. Variance
      d. Standard deviation
   3. Introduction to Normal Distribution
      a. Bell-curve
      b. Relation to standard deviation

PROBABILITY

Development

Previously students have studied empirical probability; the counting principle; permuations and combinations (notation and formuals); and theoretical probability (simple and compound).
In this unit, the main topic is the binomial theorem and its application to probability and statistics.

A. Probability
   1. Review of previous work with permuatations, combinations and probability.
   2. Extension of techniques

a. Simple conditional probability

An informal approach to conditional probability can be demonstrated using a sample space and selecting the appropriate outcomes as in the following:

What is the probability that if letters from the word MASH are chosen at random to form 4 letter words:

a. The word MASH is spelled?

b. The word MASH is spelled, knowing that the first letter chosen was an "M"?

c. The word MASH is spelled, Knowing that the first letter chosen is "s"?

b. Combinatorics applied to probability

Improving the students problem solving techniques in this area may be accomplished by taking relatively simple problems and introducing new methods, such as...

From a pack of playing cards, 4 cards are drawn. What is the probability of obtaining 1 card from each suit?

This may be considered as:

$$P(H,D,S,C \text{ in any order}) = P(H)\cdot P(D)\cdot P(S)\cdot P(C)\cdot 4!$$

$$= \frac{13}{52}\cdot\frac{13}{51}\cdot\frac{13}{50}\cdot\frac{13}{49}\cdot 4! = .1055$$

OR

$$P(H,D,S,C \text{ in any order}) = \frac{{}_{13}C_1 \quad {}_{13}C_1 \quad {}_{13}C_1 \quad {}_{13}C_1}{{}_{52}C_4} = \frac{2197}{20825} = .1055$$

c. Probability of r successes in n trials

Consider the following problem:

Example: A fair die is tossed 3 times. What is the probability of obtaining 2 heads and 1 tail?

B. The Binomial Expansion

Many experiements subject to probability analysis involve independent repeated trails of the same experiment. When each of the trails has only two mutually exclusive outcomes(head/tails;defective/nondefective; etc.) then the binomial expansion becomes a very useful tool in calculatin the probabilities involved.

Example 1: Find the 5th term of the expansion of $(a + b)^4$

Example 2: Referring to the "coin" problem of section A of this Unit? We may think of this as repeated trials(flipping one coin 3 times). Each time we get either a head or a tail, and we may write this as:

$$(H + T)^3$$

Expanding this we have: $1H^3 + 3H^2T + 3HT^2 + 1T^3$

This expression is a summary listing all possible outcomes of the given experiment. For example, the specific outcome of 2 heads and 1 tail would be given by the 2nd term: ${}_3C_2H^2T^1$ which says that this event may

Happen in 3 different ways: HHT, HTH, THH. It can also be viewed as 2 successes in 3 trails: ${}_3C_2[P(H)]^2 \cdot [P(T)]^1$. Since $P(H) = P(T) = \frac{1}{2}$ then the solution to the given problem is ${}_3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = 3/8$.

STATISTICS

Development

In Course I, the students have studied measures of central tendency, frequency histograms, and cumulative frequency histograms. There is no Statistics unit in Course II.

In this unit, the statistics studied includes sigma notation, measures of central tendency of grouped data, measures of dispersion, and introduction to the normal curve.

- Statistics

1. Measures of Central Tendency
   a. Review of previous work
   b. Sigma Notation
   c. Extension to include grouped data

In Course I much work was done finding group modes and medians not group means. The following illustrates the usual methods for organizing and studying data which is grouped

Example: Astronauts landing on planet Zap recorded the following Zapian heights, in cm, of the strange creatures they encountered:

9.3	9.2	9.3	9.3	8.9	9.0	8.8	8.4	8.4	8.1	8.1
8.2	8.2	7.7	7.8	7.8	7.4	7.3	7.4	7.4	7.5	6.9
6.9	6.5	6.5	6.2	6.2	6.2	5.8	5.8	5.3	5.3	5.5	5.5
4.8	4.8	4.8	4.4	4.4	4.4	4.0				

Summarizing this data

Note: $\sum F_i = 41$

$\sum x_i \cdot F_i = 283.7$

x_i	F_i Frequency	$x_i \cdot F_i$	x_i	F_i	$x_i \cdot F_i$	x_i	F_i	$x_i \cdot F_i$
9.3	3	27.9	8.1	2	16.2	6.5	2	13.0
9.2	1	9.2	7.8	2	15.6	6.2	3	18.6
9.0	1	9.0	7.7	1	7.7	5.8	2	11.6
8.9	1	8.9	7.5	1	7.5	5.5	2	11.0
8.8	1	8.8	7.4	3	22.2	5.3	2	10.6
8.4	2	16.8	7.3	1	7.3	4.8	3	14.4
8.2	2	16.4	6.9	2	13.8	4.4	3	13.2

So for the data viewed in this way we conclude: The modes are: 9.3 7.4, 6.2, 4.8, 4.4 (not too definitive)!

The median is 7.3

and the mean is found using the formula: $\bar{x} = \frac{\sum_{i=1}^{n} (x_i F_i)}{\sum_{i=1}^{n} F_i}$

in this case: $\bar{x} = \frac{283.7}{41} = 6.9$

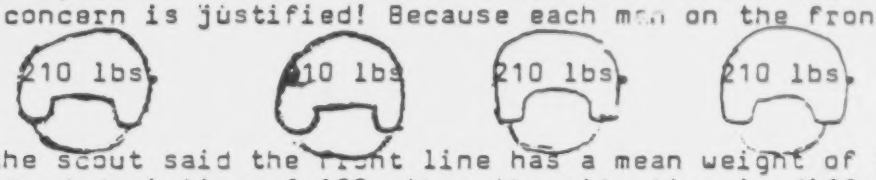
Measures of dispersion and introduction to the normal curve

Information about the variability of data gives one studying it a valuable tool for predicting and forming conclusions. The most important variability statistic is the standard deviation (s.d), and its associated statistic, the variance, which is the square of the standard deviation.

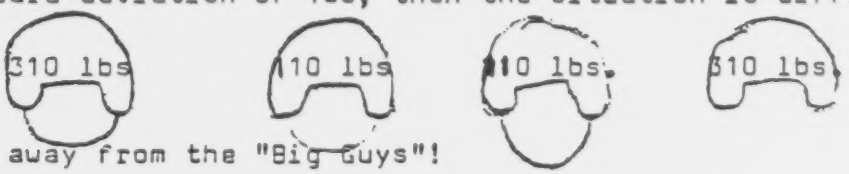
The standard deviation formula we will use :

$$\text{s.d.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\bar{x}-x_i)^2}$$

Example 1: A high school football scout reports to his coach findings concerning next week's opponent. If he tells him that the opposition's front line has a mean weight of 210 lbs., the coach has cause for some concern.
However, if in addition he was told that the standard deviation was 0, then the concern is justified! Because each man on the front line is 210lbs.



If the scout said the front line has a mean weight of 210 lbs. and a standard deviation of 100, then the situation is different.



Stay away from the "Big Guys"!

Knowing the standard deviation gave the coach valuable practical information.

Example 2: Referring back to our Zapian friends (on planet Zap) in section B of this unit $\bar{x} = 6.9$; $n = 41$

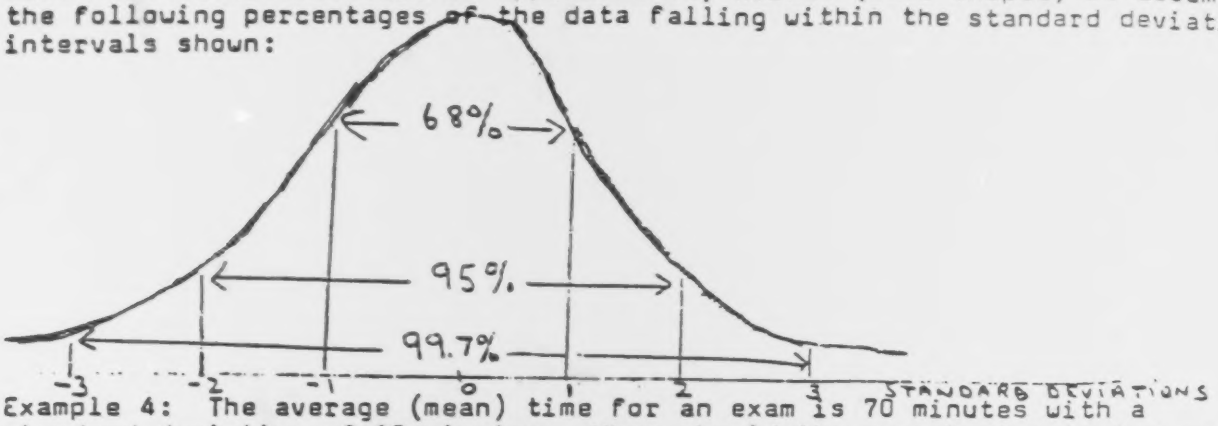
Therefore: $\text{s.d.} = \sqrt{\frac{1}{41}(6.9-9.3)^2 + (6.9-9.2)^2 \ldots}$

$= \sqrt{\frac{1}{41}(102.52)}$

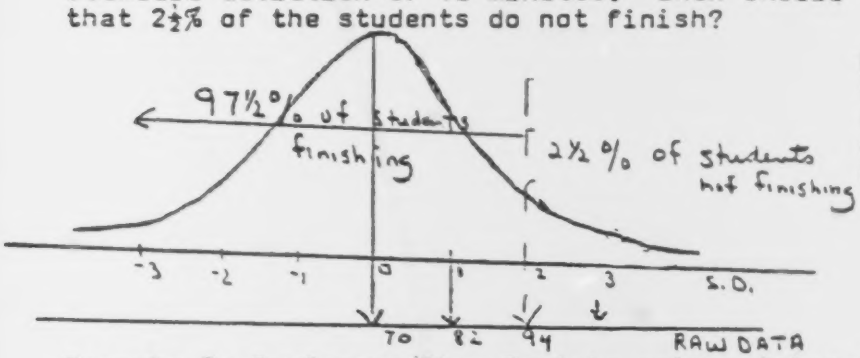
$= \sqrt{2.5} \sim 1.58$

This means that our little extraterrestrial friends in addition to being short do not vary too much in height.

Example 3: A set of measurements whose distribution is "bell" shaped-more formally called a normal distribution is very useful to the statistian. Given that a distribution is approximately normal (bell-shaped) we assume the following percentages of the data falling within the standard deviation intervals shown:

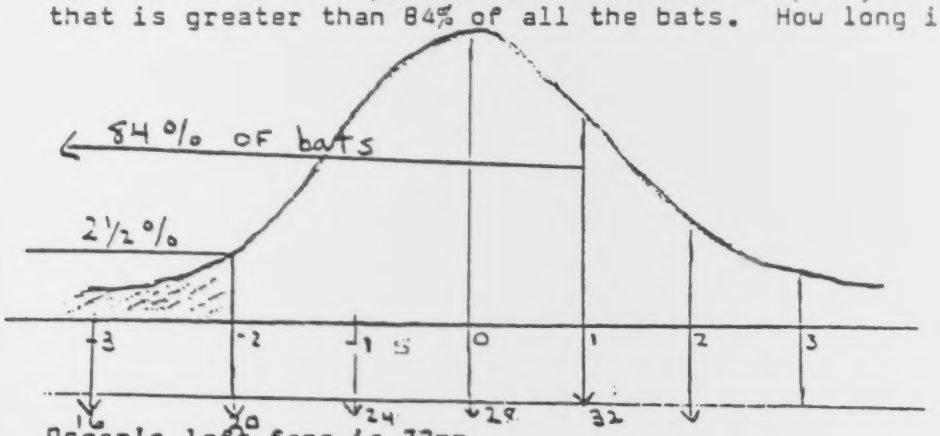


Example 4: The average (mean) time for an exam is 70 minutes with a standard deviation of 12 minutes. When should the exam be terminated so that 2½% of the students do not finish?



From our analysis, we see that if the exam is terminated after 94 minutes 2½% of the students will not finish the exam.

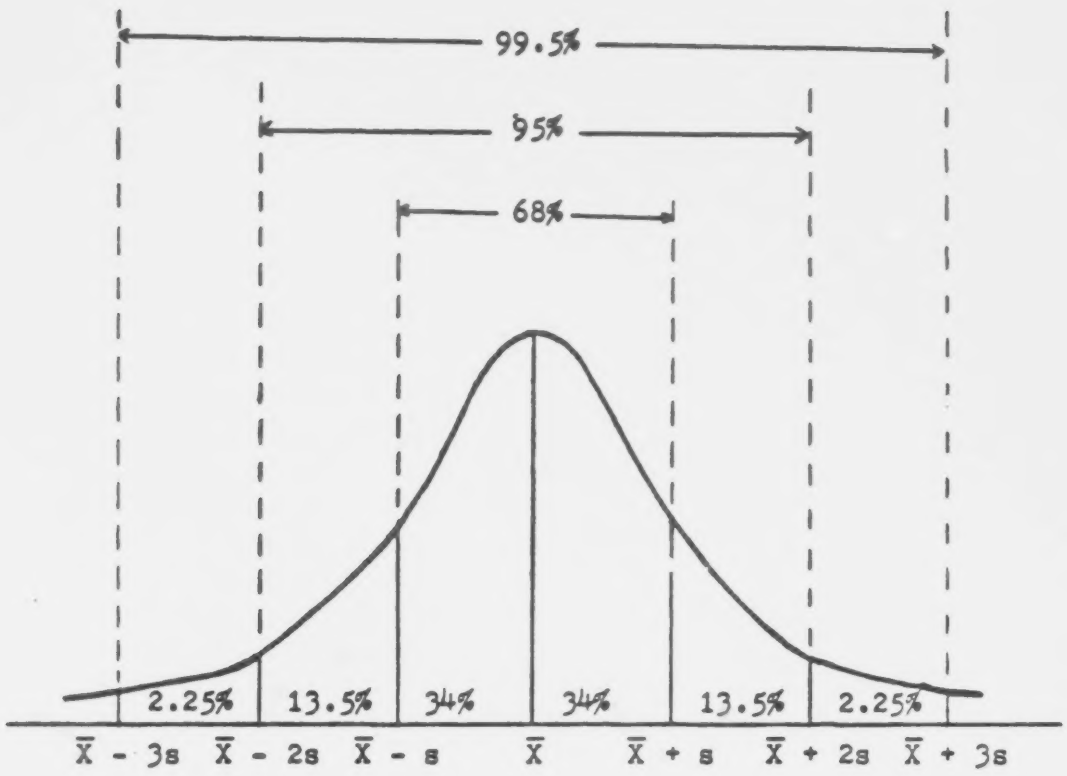
Example 5: Professor Stan Deviate raises laboratory vampire bats. The length of their left fangs are normally distributed with a mean of 28 mm and a s.d. of 4 mm. The professors's favorite vampire, Oscar, has a left fang that is greater than 84% of all the bats. How long is Oscar's left fang?



Oscar's left fang is 32mm

Example 6: What is the probability that one of the professor's bats has a left fang less that 20mm? According to the diagram above, 2½% of the bat population falls into this category. Therefore, the probability is .025.

STANDARD DEVIATION CURVE (BELL CURVE)



PERCENTILE 2.5 16 50 84 97.5

1. If a "fair"die is tossed, what is the probability that:
   a. a 3 will appear.
   b. a 1 will appear.
   c. an even number will appear.
   d. a number greater than 4 will be otained.
   e. a 7 will appear.
   f. a number less than 7 is shown.

2. In a used car lot there are 2 green hardtops, 4 red convertibles, 5 blue hardtops, 3 red hardtops, 4 blue convertibles and 2 green convertible In his attempt to get away from the police, a certain bank robber decided he needed a get away car. Stealing a car would be too risky, so he went to the used car lot. Since the color or the model made no difference to him , the car dealer chose a car for the sale. What is the probability that the dealer selected a car that was: a. red? b. a convertible? c. red or a convertible? d. blue and a hardtop? e. What is the probability that the bank robber will make it out of the state?

3. Two coins, a nickel, and a dime are tossed. What is the probability of obtaining: a. two heads? b. a head on the nickel and a tail on the dime? c. a head and a tail? d. at least one head? e. exactly one head? f. two tails? g. a head on the first toss?

5. A coin is tossed three times. What is the probability of obtaining a. three heads b. heads on the first two tosses and a tail on the third toss c. two heads and a tail in any order d. a tail on the second toss e. tails on the first two tosses?

6. A jar has 7 red marbles, 3 green marbles, and 5 blue marles. A marble is picked at random and then put back and a second marble is drawn. What is the probability of drawing: a. two green marbles? b. a blue marble and then a green marble? c. a blue and a green marble in any order? d. two the same color?

7. Calcualate the answers to problem 6 if the first marble is not replaced.

Worksheet 2

1. Town A is connected to town B by 6 roads. Town B is connected to town C by 5 roads. If you wish to travel from town A to twon C via town B, how many different routes may you choose?

2. How many different "words" of 6 letters each can be formed from the letters of the word "HYPERBOLA"/

3. The library contains 300 books. If you wish to borrow 3 books, how many ways may you make your choice?

4. A group of 7 boyscouts is camping out. They have a tent that can accommodat 3 boys and one that accommodates 4. In how many ways may the boys arrange themselves?

5. Charlie is getting ready for a big weekend. He plans to have a date for Fri. Sat. and Sun. nights. So he takes out his little black book and recalls 73 names in it. How many choices for dates over the weekend does he have if
   a. he will not take one girl out more than once on any given weekend?
   b. one girl may be asked out any number of times?
   c. the order in which he asked the girls matters?

6. How many 3 digit numbers less than 700 can be formed from the digits 3,4,5,6,7,8,9?(no repetition allowed)?

7. In how many ways can a class elect a president, vice-president, secretary, and treasurer from a class of 100?

8. In how may ways can 8 people be seated in a rou of eight chairs if 2 pecple insist on sitting next to each other?

9. How many license plates can be made using any two letters and any three digits(repetition allowed)?

10. How many ways may a committee of 4 be chosen from a group of 25?

11. From a goup of 25 Democrats and 18 Republicans, how many committees consisting of 3 Dems and 2 Reps are possible?

Worksheet 2 (cont'd)

12. In how many ways can a person get a bridge hand which consists of 3 aces, 1 king, 2 jacks, and 7 other cards?

13. Find the number of ways of arranging the letters in the word Cincinnatti?

14. In how many different ways may 5 white flags and 3 blue flags be arranged one above the other?

15. There are 2 paths in going from the point (1,1) to the point (2,2) on a coordinate plane. We can either: 1. Move up one unit (u) and to the right one unit (r) or, 2. Move to the right one unit (r) and then up one unit (u). Show there are 35 such paths in going from the point (1,1) to the point (5,4) if we can make on a u or r decision at each point (x,y) where x and y are integers.

Worksheet 3

1. A committee of 6 is chosen from 8 men and 10 women. What is the probability that the committee consists entirely
   a) of men ?
   b) of women ?

2. A bridge hand is made up of 13 cards.
   a) How many possible "bridge hands" exist?
   b) How many different ways could you be dealt a hand consisting of 3 kings 2 queens, 2 aces, and rest other kinds of cards?
   c) What is the probability of being dealt this "hand"?

3. An urn contains 10 marbles: 5 black, 2 green, 3 yellow
   When drawing three marbles, what is the probability that a) all 3 are black b) all 3 are yellow? c) all 3 are green?

4. From a deck of 52 cards, 2 are chosen at random. What is the probability that both cards are hearts?

5. A coin is tossed 100 times. What is the probability of getting exactly 47 heads?

6. A loaded coin has p(Head) = 2/3 It is tossed 12 times. What is the probability of not more than 8 heads?

7. You enter a room and find two strangers there. What is the probability that at least two of the three of you were born in the same month?

8. Two equally matched teams play in the World Series. The first to win 4 games is the victor. If team A is behind 1 game to 3 games. What is the probability that team wins the series?

9. There are two urns, A and B. Urn A contains one black and one red ball. Urn B contains two black and tree red balls. A ball is chosen at random from urn B. What is the probability that both balls are the same color?

"Problems in probability, etc.

1. If there are five white socks and seven black socks in a drauer, and I must remove them without first seeing them (the room is dark) what is the probability that I will get a pair if I take three socks?

2. Three cards are usec for this probability question. The cards are as follows:

   Card 1 Red front, Red back

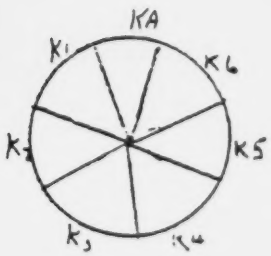
   Card 2 Red front, white back

   Card 3 White front, white back

   The cards are shuffled out of view of the class and one card is chosen and placed on a table so that only one side is seen. If the one side that can be seen is red what is the probability that the other side is red?

3. How could I figure out how many fish are in a lake?

4. One afternnon, King Arthur and 6 knights which we will refer to as $K_1, K_2, K_3, K_4, K_5, K_6$ were going to get together for a round table discussion. The seating arrangement of the knights and King Arthur when they sat down is given by the diagram below. K_6 offered the following bet to King Arthur: "Each time that we have one of these discussions we all sit in this same order. Suppose that we sat in a different order each time. I'll bet you 1000lbs. in gold that there are more than 5000 ways that we could be arranged at this round table". Should King Arthur take this bet?



GRAPHS, PROBABILITY, STATISTICS

Topic 12

0.12.01 Number relationships

The student will be able to...

- *construct a bar graph using blocks/cubes
- *transfer a graph to a pictorial representation
- *interpret pictorial representations as those representing concrete objects or people
- *identify number relationships using symbols and pictures
- *count aloud the number of squares that are colored on a bar graph
- *color the corresponding number of squares on a vertical bar graph when given concrete objects or the pictorial representation
- *write the numeral that represents the number of squares colored on a vertical and horizontal bar graph

0.12.02 Graph reading

The student will be able to...

- *construct and interpret a simple picture graph

0.12.03 Chance events/predicting

The student will be able to...

- *distinguish between certain and uncertain in regard to events
- *distinguish between more or less likely in chance situations
- *recognize the outcomes of activities as chance (guessing)

1.12.01 Pictographs

The student will be able to...

- *collect, organize and graph data relating to preference using two categories at a time (A)
- *read and interpret the information given in a graph using more than two categories (A)
- *use information on a graph to answer questions involving more or less (A)

1.12.02 Bar graphs

The student will be able to...

- *color the corresponding number of squares on vertical or horizontal bar graphs, when given concrete objects or the pictorial representation (B)
- *identify and record the number of squares that are colored on a bar graph (B)
- *gather, organize and record data based on personal or group experience to complete a bar graph using one-to-one correspondence (B)

1.12.03 Probability - chance events

The student will be able to...

- *predict the likelihood of events happening (B)
- *record the outcome of a chance event (B)

2.12.01 Data - collect, tabulate, arrange and compare in tables

The student will be able to...

- *collect, tabulate and arrange data in a table (A)
- *compare data in a table (A)

2.12.02 Bar graphs

The student will be able to...

- *collect and structure data based on personal or group experiences to complete a bar graph (A)
- *interpret a bar graph (A)

2.12.03 Probability - predict and tally outcomes

The student will be able to...

- *predict the ourcome of three equally likely events (B)
- *tally the outcome in a three-possibility event (B)

2.12.04 Arrangements of 2 through 5 items

The student will be able to...

- *make arrangements of 2 through 5 items using all the items (B)
- *define and identify subsets of a given set of objects (B)
- *make subsets of 1 or 2 objects from sets of 3 or 4 concrete objects (B)

2.12.05 Ratio - 2:1 correspondence

The student will be able to...

- *use two-to-one correspondence to develop the concept of ratio (B)
- *use a chart to show ratio (B)

3.12.01 Bar graphs

The student will be able to...

- *read and interpret a bar graph (A)
- *construct a bar graph (A)

3.12.02 Probability

The student will be able to...

- *estimate (predict) the outcomes of events, and compare the results with the predictions (A)
- *predict the outcome of unequally likely events (A)

3.12.03 Arrangements - tree diagrams

The student will be able to...

- *list and count the number of possible arrangements in a set of objects (A)
- *make a tree diagram to display arrangements of given data (A)

4.12.01 Graphs

The student will be able to...

- *make frequency tables from tallied data (A)
- *graph data drawn from newspapers, magazines, polls, and appropriate activities (A)
- *read and interpret information from a line graph (B)
- *construct a line graph from a table of data (B)

4.12.02 Statistics

The student will be able to...

- *find the mean (average) from given data (A)
- *solve word problems involving finding the averages of given numbers (A)
- *define the range as the difference between the largest and the smallest numbers in a set (A)
- *identify the mode as the number that occurs most frequently (B)

4.12.03 Probability

The student will be able to...

- *determine probabilities by comparing favorable outcomes with total possible outcomes and express as fractions (A)
- *determine probabilities where the outcomes are not equally likely and express as fractions (A)
- *predict the outcomes of probability experiments (A)

4.12.04 Arrangements

The student will be able to...

- *identify all the possible arrangements for a group of objects (A)
- *solve word problems related to skills learned in this module (A)

5.12.01 Graphs

The student will be able to...

- *construct and interpret a double bar graph (B)
- *choose the appropriate graph to show information (B)

5.12.02 Statistics -mean and median

The student will be able to...

- *compute the arithmetic mean (average) to the nearest whole number (A)
- *recognize that the median is the middle score when the group of scores is arranged according to size (limit to odd number of scores) (B)
- *analyze collected data according to range and means (A)

5.12.03 Circle graphs

The student will be able to...

- *interpret circle graphs when the sectors represent fractional numbers with unlike denominators (A)
- *construct circle graphs where the sectors represent fractional numbers with unlike denominators (A)

5.12.04 Probability

The student will be able to...

- *identify events that are impossible, certain, and that sometimes occur (A)
- *express probability as a fraction (A)
- *recognize that some experiments have outcomes that are not equally likely (A)
- *represent and count the number of elements in the sample space of an experiment (A)
- *conduct probability experiments using an increasing number of trials so that the results become more reliable (A)

5.12.05 Algebra - ordered number pairs

The student will be able to...

- *plot an ordered pair on a graph (A)
- *solve equations with two variables to find ordered pairs and graph the results (B)

6.12.01 Graphs

The student will be able to...

- *interpret and construct circle graphs from a table of data (A)
- *construct and compare bar, line and circle graphs and note the advantages and disadvantages of each (A)

6.12.02 Statistics

The student will be able to...

- *find the mode of a given set of data (A)
- *find the median of a given set of data (A)
- *identify the mean, median and mode after involvement in an extended recording project (A)

6.12.03 Compound Probability

The student will be able to...

- *determine the probability of one outcome from a sample space (A)
- *determine the probability of two independent events occurring together from a sample space (A)

6.12.04 Coordinate systems

The student will be able to...

- *locate, read, and interpret coordiantes on a graph (C)
- *graph points whose coordinates include positive and negative numbers using all 4 quadrants (C)

6.12.05 Permutations and combinations

The student will be able to...

- *count the number of permutations that can be made with a set of objects (A)
- *use the multiplication principle to determine the number of permutations that can be made with a set of objects
- *count the number of groups when making combinations (A)

7.12.01 Probability

The student will be able to...

- *construct a sample space for various experiments (A)
- *read and use the expression P(A) to represent the probability of event A (A)
- *write the probabilities involved with simple experiments: flipping a coin, rolling a die (A)

7.12.02 Probability with a spinner

The student will be able to...

- *state that the sum of the probabilities in a given experiment is always 1 (A)
- *state and use the meaning of equally likely events (A)
- *state and use the meaning of complementary events to their probabilities (A)

7.12.03 Multiple graphs

The student will be able to...

- *construct, read and interpret a multiple bar graph (involving a two bar comparison) (B)
- *read and interpret a multiple line graph (involving a two line comparison) (B)
- *construct, read and interpret a circle graph (involving fractional or percent comparisons) (B)

7.12.04 Frequency tables

The student will be able to...

- *obtain the mode from the frequency table (A)
- *recognize how changes in data affect the mean, median and mode (A)

7.12.05 Compound probability

The student will be able to...

- *determine the probability of two independent events occurring together by using a sample space (B)
- *determine the probability of either of two outcomes occurring by using a sample space (B)

8.12.01 Multiple graphs

The student will be able to...

- *construct, read and interpret a multiple bar graph (involving a two bar comparison) (A)
- *construct, read and interpret a multiple line graph (involving a two line comparison) (A)
- *construct, read and interpret a circle graph (involving fractional or percent comparisons) (A)

8.12.02 Statistics

The student will be able to...

- *read, interpret and make a histogram (A)
- *make a frequency polygon from a histogram (A)

8.12.03 Comparison of permutations and combinations

The student will be able to...

- *determine a permutation of n things taken r at a time where $r \leq n$ (A)
- *compare the number of arrangements in a permutation with the number of arrangements in a combination (A)

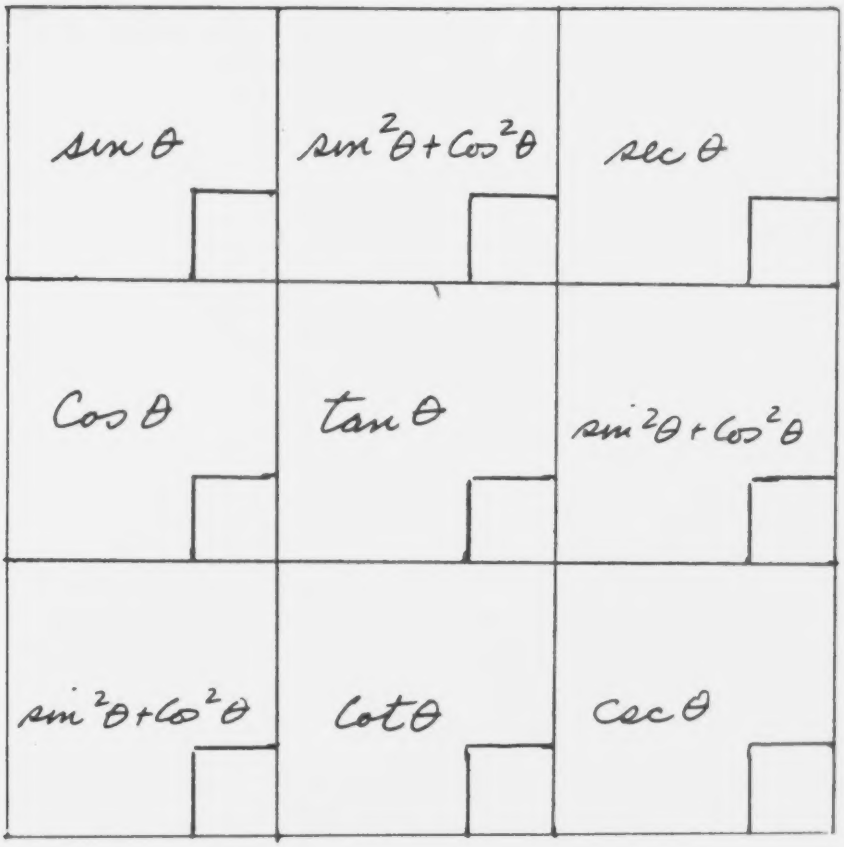
8.12.04 Compound probability

The student will be able to...

- *determine the probability of two independent events occurring together by using a sample space and a formula (A)
- *determine the probability of either of two outcomes occurring by using a sample space and a formula (A)
- *determine the probability of more than two independent outcomes occurring at the same time using a formula (B)
- *determine the probability of exactly, at least or at most r successes in n trials by experimentation (B)

TEACHING LOGARITHMIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS

TRIG - TAC - TOE I



RED & BLACK CUBES

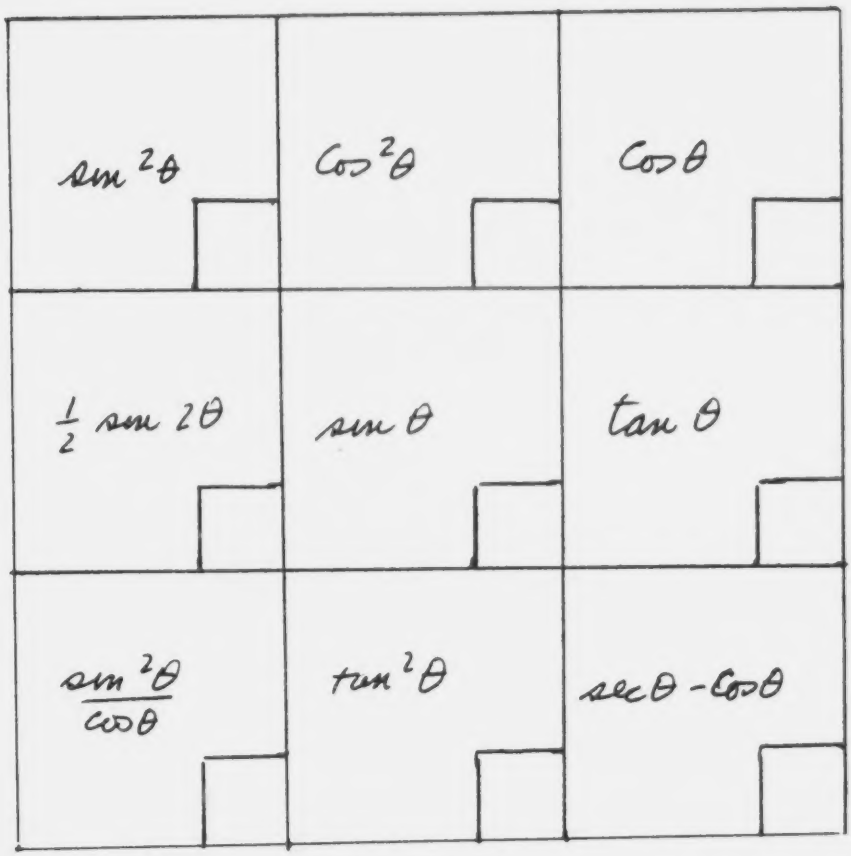
RED ÷ BLACK

TRIG - TAC - TOE II

$\sin^2\theta + \cos^2\theta$	$\frac{1}{2}\sin 2\theta$	$\sin\theta$
$\sin^2\theta$	$\frac{1}{2} - \frac{1}{2}\cos 2\theta$	$\cos\theta$
$\cos^2\theta$	$\frac{1}{2} + \frac{1}{2}\cos 2\theta$	$\frac{1}{2}\sin 2\theta$

RED & BLACK CUBES MULTIPLIED

TRIG - TAC - TOE III



BLUE & GREEN CUBES

BLUE ÷ GREEN

TRIG - TAC - TOE IV

1	$\csc\theta$	$\sec\theta$
$\cot\theta$	$\tan\theta$	$\sin^2\theta + \cos^2\theta$
$\cos\theta$	$\frac{\sin\theta}{\cos^2\theta}$	$\frac{\cos^2\theta}{\sin\theta}$

BLUE & GREEN CUBES
MULTIPLY

GRAPHS OF TRIGONOMETRIC FUNCTIONS:

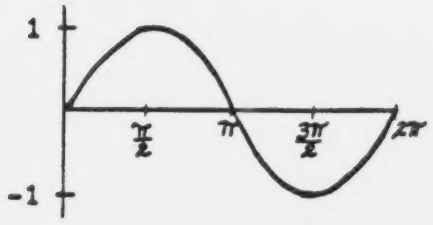
a) $y = a \sin bx$

a = amplitude (maximum & minimum values)

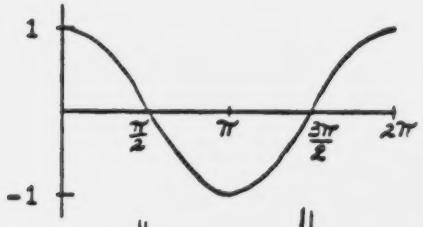
b = frequency (number of complete curves seen from 0 to 2π)

period = $\frac{2\pi}{b}$

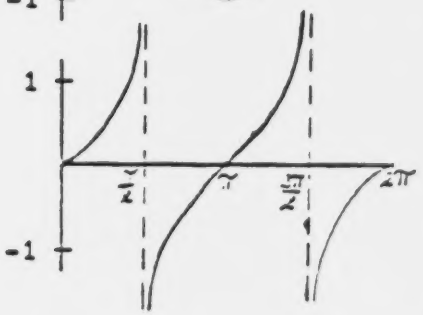
b) $y = \sin x$



c) $y = \cos x$



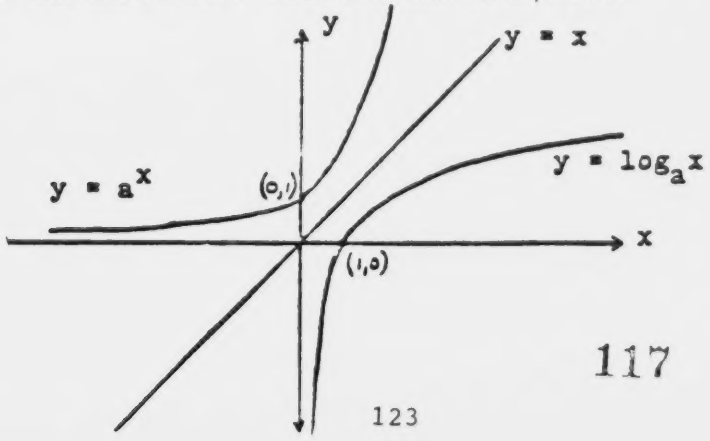
d) $y = \tan x$



EXPONENTIAL VS. LOGARITHMIC FORM

1. $x^a = b \longleftrightarrow \log_x b = a$

2. GRAPHS OF EXPONENTIAL & LOGARITHMIC FUNCTIONS:



TRIGONOMETRY

Part I: Answer all questions in this part show all work. (5 credits each)

1. Express tan (-140°) as a function of a positive acute angle.

2. In how many points do the graphs of $y = \frac{1}{2}$ and $y = \sin x$ intersect if they are drawn on the same set of anxes, where the domain of x is $\{0° < x < 360°\}$?

3 Find the value of $(\cos \frac{\pi}{6}) \cdot (\sin \frac{\pi}{2})$ in simplest radical form.

4. In which quadrant are both cotangent and cosecant negative?

5. Express 240° in radian measure.

6. $\cos x = \sin (x - 20°)$. A value of x may be (A) 55° (B) 70° (C) 90° (D) 100°

7. $\cos 262° = ?$ (A) $\sin 8°$ (B) $-\cos 8°$ (C) $-\sin 8°$ (D) $\cos 8°$

8. $\sec^2 x + \csc^2 x = ?$ (A) $\sin^2 x \cos^2 x$ (B) $\frac{1}{\sin^2 x \cos^2 x}$
(C) $1 + \tan^2 x$ (D) $1 - \tan^2 x$

9. $\sin 48° = ?$ (A) CSC 42° (B) COS 42° (C) CSC 48° (D) COS 48°

10. $\frac{\cos^2 x}{\sin x} + \sin x = ?$ (A) 1 (B) $\frac{1}{\csc x}$ (C) $\frac{1}{\sin x}$ (D) $\cos^2 x$

11. The smallest positive value of x which satisfies $2\cos^2 x + \cos x - 1 = 0$ is (A) 0 (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D $\frac{\pi}{4}$)

12. $\frac{2(1 + \cos A)}{\sin^2 A + \cos A + \cos^2 A} = ?$ (A) 1 (B) 2 (C) $\frac{2}{\sin A}$ (D) $\frac{2}{\cos A}$

13. If $\sin^2 x = 1$, x may equal (A) $\frac{3\pi}{2}$ (B) 0 (C) π (D) $\frac{\pi}{4}$

14. Express, in degrees, the measure of the positive acute angle θ, which satisfies the equation $2 \sin \theta - 1 = 0$

15. $\frac{\text{Tan } \theta}{\text{Sec } \theta}$ = (A) Cos θ (B) $\frac{\text{Sin } \theta}{\text{COS}^2 \theta}$ (C) $\frac{\text{COS}^2 \theta}{\text{SIN } \theta}$ (D) Sin θ

Part II Show all work (10 Points)

16. Solve for 0 if $0° \leq 0 \leq 360°$

$2 \text{ Cos}^2 \theta = 3 - 3 \sin \theta$

17. Prove the identity (15 points)

$\frac{\text{Cot } x \sec^2 x}{1 + \text{col}^2 x} = \tan x$

STANDARDS FOR RATING REGENTS

EXAMINATIONS

35. *Question*

The coordinates of the vertices of $\triangle ABC$ are $A(1,6)$, $B(2,9)$, and $C(7,10)$.

a Graph and label $\triangle ABC$. [1]

b Graph and state the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ after a reflection over the line $y = x$. [3]

c Graph and state the coordinates of $\triangle A''B''C''$, the image of $\triangle A'B'C'$ after a reflection over the x-axis. [3]

d Graph and state the coordinates of $\triangle A'''B'''C'''$, the image of $\triangle A''B''C''$ after the transformation $(x,y) \rightarrow (x-5,y+3)$. [3]

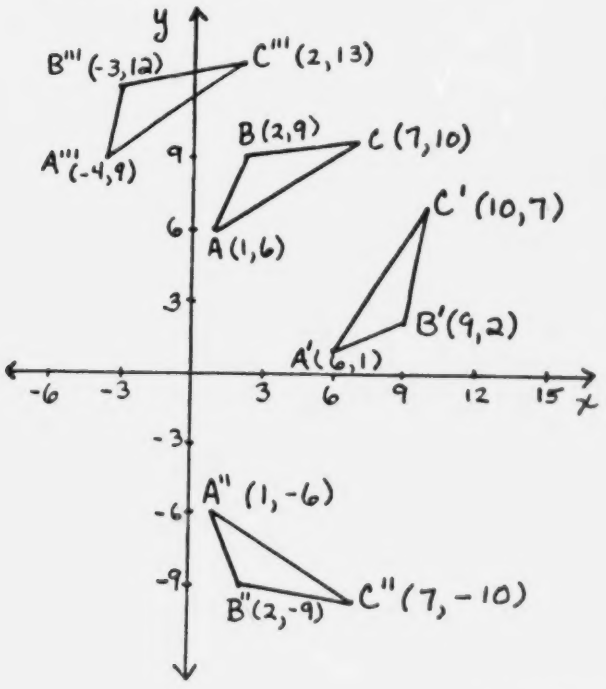
Correct Solution

b $A'(6,1), B'(9,2), C'(10,7)$

c $A''(6,-1), B''(9,-2), C''(10,-7)$

d $A'''(1,2), B'''(4,1), C'''(5,-4)$

Student Response



(+9 credits)

Rating Explanation

a Correct (+1)

b Correct (+3)

c Use of $\triangle ABC$ as pre-image, instead of $\triangle A'B'C'$, is not a cardinal error in principle. (+2)(-1) [GEOM-10]

d $\triangle ABC$ was again used as the pre-image; however a deduction was already made in part *c*, and the student should not be penalized twice for the same mistake. (+3) [GEOM-10; GEN-12]

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Notes

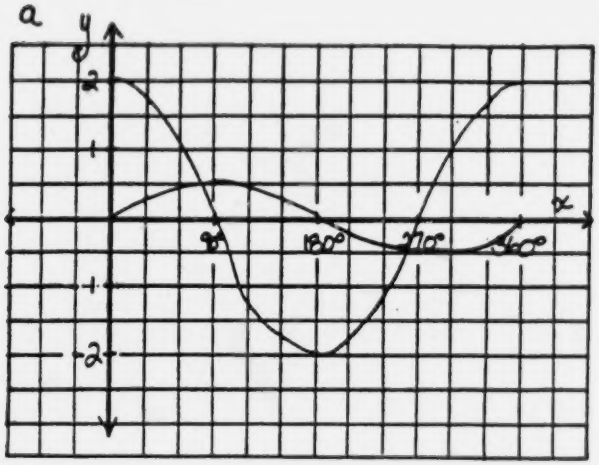
18. *Question*

a On the same set of axes, sketch and label the graphs of $y = \sin \frac{1}{2}x$ and $y = 2 \cos x$ as x varies from $-\pi$ to π radians. [8]

b From the graphs made in part *a*, how many values of x in the interval $-\pi \leq x \leq \pi$ satisfy the equation $\sin \frac{1}{2}x = 2 \cos x$? [2]

Student Response

a



b 2

28. *Question*

Find to the *nearest degree*, all values of θ in the interval $0° \leq \theta \leq 360°$ which satisfy the equation $3 \cos 2\theta + \sin \theta - 2 = 0$. [10]

Student Response

$3\cos 2\theta + \sin\theta - 2 = 0$

$3(1 - 2\sin^2\theta) + \sin\theta - 2 = 0$

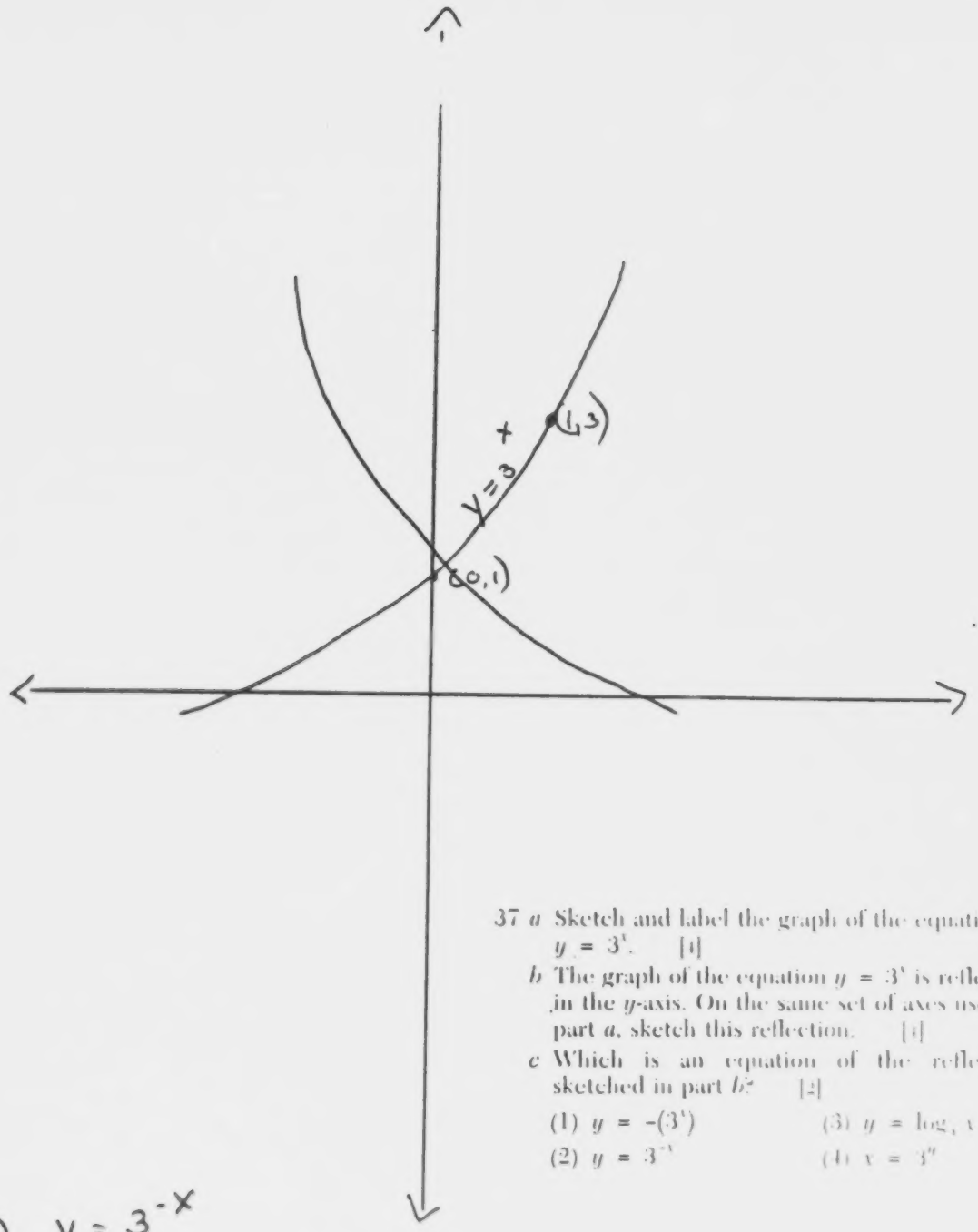
$-6\sin^2\theta + \sin\theta + 1 = 0$

$6\sin^2\theta - \sin\theta - 1 = 0$

$(3\sin\theta - 1)(2\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{3}$ $\quad$ $\sin\theta = -\frac{1}{2}$.

$20°, 160°$ $\quad$ $210°, 330°$



37 *a* Sketch and label the graph of the equation $y = 3^x$. [4]

b The graph of the equation $y = 3^x$ is reflected in the *y*-axis. On the same set of axes used in part *a*, sketch this reflection. [1]

c Which is an equation of the reflection sketched in part *b*? [2]

(1) $y = -(3^x)$ (3) $y = \log_3 x$

(2) $y = 3^{-x}$ (4) $x = 3^y$

c) (2) $y = 3^{-x}$

13. *Question*

Find, to the *nearest tenth*, the standard deviation for the following data. [8]

mean (x_i)	frequency (f_i)
20	3
22	2
23	2
25	2
30	1

Student Response

x_i f_i $x_i - \bar{x}$ $(x_i - \bar{x})^2$

20	3	-4	16
22	2	-2	4
23	2	-1	1
25	2	1	1
30	1	6	36

$\frac{120}{5} = 24$ $\frac{58}{5} = 11.4$

S.D. $= \sqrt{11.4} = 3.38$

$\frac{1.0569}{2} = .5284$

40 Using the following set of data, find the:

a mean [2]
b median [1]
c mode [1]
d standard deviation to the *nearest tenth* [6]

Measure (x_i)	Frequency (f_i)
70	3
75	4
80	7
85	6

18 The scores of an exam have a normal distribution. The mean of the scores is 48 and the standard deviation is 5. Approximately what percent of the students taking the exam can be expected to score between 43 and 53?

(1) 95% (3) 34%
(2) 68% (4) 13%

36 *a* Find, to the *nearest tenth*, the standard deviation for the following data. [8]

Measure (x_i)	Frequency (f_i)
20	4
21	2
24	5
26	3
30	6

b The above data do *not* form a normal distribution. If a measure is selected at random from the above data, what is the probability that it will differ from the mean by less than one standard deviation? [2]

GUIDE FOR RATING

REGENTS EXAMINATIONS IN MATHEMATICS

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THE STATE EDUCATION DEPARTMENT
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April 1986

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INTRODUCTION

The purpose of this guide is to provide a set of directions, together with some examples, to assist teachers in the rating of answer papers for Regents examinations in mathematics. While it is not possible to anticipate all the possible questions that may arise, the suggestions and examples in this guide deal with those that tend to occur most frequently.

In all work in mathematics the aim should be accuracy – not only in the mechanical aspects but also in the aspects that require reason, judgment, and application. In every instance where an error occurs, to make proper deductions, a careful consideration should be given to the relative amount and importance of these aspects in the problem.

The principal is responsible for establishing rating procedures that will assure reasonable confidence in the accuracy of the scores assigned to the Regents answer papers by individual teachers or by committees of teachers. The criterion in all cases shall be that the rating assigned to a student's paper shall be a fair and accurate rating of that paper.

The scoring key provided by the Department is the official key and teachers must rate according to this key. Unless otherwise specified, mathematically correct variations in the answers are allowed. It should be understood that any special ruling or rating as indicated on the Regents examination or scoring key takes precedence over any suggestion made in this guide and is not considered as conflicting. The staff of the Bureau of Mathematics Education is available to answer questions concerning the rating of Regents examination papers. Call or write Fredric Paul, Chief, Bureau of Mathematics Education, New York State Education Department, Albany, New York 12234 (518: 474-3900).

The following rating suggestions are divided into seven sections: general, algebra, geometry, graphing and coordinate geometry, circular functions, logic, and probability and statistics. These sections are followed by typical Part I, II, and III examination questions that have been rated to show the appropriate credit allowed. Each rated question has an explanation of that rating. Many of the rating guidelines are referenced by number to questions in the example section. Each question is, in turn, referenced to one or more of the rating guidelines using GEN, ALG, GEOM, GRAPH, CIR FN, LOGIC, or PROB-STAT.

16. Partial credit may be given for incomplete solutions of problems or exercises when it is clear that the method used could be carried on to a correct conclusion. In such cases, however, the work should be carried far enough to indicate clearly that the student is certain of the procedure and that failure to finish the problem is due either to lack of time or to the inability to see the next step in the solution. The amount of credit allowed depends upon the amount and degree of importance or difficulty of the work that has been done correctly. No credit should be given for merely enumerating extraneous facts or principles. [Ex-29(a),(b)]

17. In the solution of numerical exercises, not more than 20 percent credit is allowed for writing the correct formula(s). If the formulas are given, then no credit should be allowed.

18. No partial credit is allowed for solving an incorrect equation which is considerably easier than the intended one; however, partial credit should be granted for solving incorrect equations of equal difficulty. *For that purpose, these alternative solutions should be carefully checked.* [Ex-21(iii); Ex-28; Ex-31]

19. Answers to questions involving computation should not be given full credit unless the degree of accuracy specified in the question is observed. Computations should be performed with appropriate accuracy (one decimal place more than the required answer), with results rounded as required. Failure to do so should receive a deduction of one credit. [Ex-13; Ex-36; Ex-37]

20. Numerical illustrations are not to be accepted in lieu of a proof. In general, proofs involving special cases should receive little or no credit, never more than 50 percent.

21. In the solution of equations, the solving of verbal problems, the demonstration of original geometric exercises, the proofs of trigonometric identities, etc., there should be little credit, if any, deducted for the omission of those steps which, in the judgment of the examiner, do not detract from the merit of the work. [Ex-14(iii); Ex-21(ii); Ex-22]

22. When a student's method of solution differs from that which is generally accepted, but is mathematically sound and employs acceptable techniques, no deduction should be made. [Ex-37]

23. The presence of irrelevant statements in answers to Part II or Part III questions is a continuing dilemma for raters. Correct irrelevant statements that do not seriously detract from the solution as a whole should receive no deduction, whereas incorrect irrelevant statements should not be penalized more than 20 percent. [Ex-12(d); Ex-22]

24. Answers to questions that merely call for the *reproduction* of definitions or principles, since they require little or no original thinking, should be severely rated. A definition or principle partly wrong or incomplete should receive no credit.

25. In problems involving computation by means of logarithms, a premium should be placed on accuracy. In general, not more than 40 percent of the credit should be allowed for the correct theory. For each mechanical error, one credit should be deducted. (See GEN-15.) [Ex-30]

26. Only the reference tables provided in the examination booklets should be used. In using these tables, mantissas and values of trigonometric functions should always be given to *four* decimal places, and unless otherwise specified in finding antilogarithms, numbers should be given to *four significant figures.* [Ex-2; Ex-3]

27. In solving equations graphically, when it is specified that the answer be estimated to the nearest tenth, the allotted credit for the reading should be granted if it is estimated to within one-tenth from the graph as made.

ALGEBRA

1. Acceptable equivalent answers commonly occur in examples involving:
   (a) Fractions sometimes not reduced to lowest terms [Ex-1]
   (b) Expressions involving square roots
   $\sqrt{121}$ or 11
   $\sqrt{48}$ or $4\sqrt{3}$ or $2\sqrt{12}$
   (c) Equivalent percents, decimals, and fractions
   (d) Equivalent absolute values
   (e) Numerical expressions where the order in which the indicated operations are performed has no effect on the result [Ex-9]

2. If the directions say to simplify, then all answers containing fractions should be single fractions in lowest terms, and those containing radicals should be in "simplest" radical form, i.e.,
   (a) No radicands with perfect square factors
   (b) No fractional radicands
   (c) No irrational denominators

3. Part I questions for which a student gives more information than is requested need to be examined carefully.
   For example:
   (a) If both values for the variables in a system of equations are correctly listed, either as a pair of equations or as an ordered pair, when only one value was asked for, full credit should be allowed. If the second value is not correct, no credit should be allowed. [Ex-5(i),(ii)]
   (b) If only the positive or negative root of an equation (or square root) is asked for, no credit should be allowed when additional roots of a different nature are listed. [Ex-4; Ex-7]
   (c) If an item asks for either the slope or y-intercept of a linear equation and *both* are correctly listed and clearly identified, full credit should be allowed; however, the use of m and b as labels is *not* to be construed as clearly identified. [Ex-6(i),(ii),(iii)]

4. In the setting up of equations to be used in the solution of verbal problems, partial credit may be given for the formation of incorrect equations where there is a minor error in reasoning.

5. For verbal problems worth [a, b], allow a credits for the analysis, which consists of the representation of the unknowns and the formation of the appropriate open sentence(s) from which the unknowns are to be obtained. The computation is worth b credits. [Ex-20; Ex-34(i),(ii)]

6. In verbal problems, the unknowns must be identified. (A labeled diagram or table is acceptable.) The credit for analysis of these problems should be divided to allow no more than 20 percent for representing the unknowns, with the remaining credit for the equation(s). [Ex-20; Ex-31]

7. In the solution of a verbal problem, a properly labeled diagram or chart may be considered as representing the unknowns.

8. In the analysis of a verbal problem that could be solved by an inequality, the use of an equation is not necessarily an error in principle. [Ex-34(i),(ii)]

9. A problem for which only an algebraic solution is acceptable should receive no more than 20 percent of the total allotted credit when only an arithmetic solution is shown or when only the correct answer appears. [Ex-20]

10. For verbal problems which lead to equations with multiple roots, 1 credit should be deducted when an inappropriate root has not been clearly rejected. [Ex-25(b)]

11. If a problem calls for the algebraic solution of a pair of simultaneous equations, then only an algebraic solution should receive full credit. A correct graphic solution using coordinate axes should receive no more than 50 percent. If required, a proper check would also receive the specified credit.

12. In checking by substituting particular values, no credit should be given unless it is done in the original equations. In checking systems of equations, no credit is allowed if the check involves only substitution into the equation from which the second unknown was determined. [Ex-32]

13. For a Part II question requiring more than one answer, each part should be rated by deducting 1 credit for each incorrect or omitted answer from the total credits allotted for that part. [Ex-15]

14. The expression $\{\phi\}$ may not be accepted when the answer is the null set, i.e., $\{\ \}$ or ϕ. Phrases such as "no solution" or "none" are acceptable.

15. When a question asks for a complex number to be expressed in $a + bi$ form, answers such as $\frac{5 + 6i}{3}$ or $\frac{1}{3}(5 + 6i)$ are not acceptable.

GEOMETRY

1. In all construction problems, accurate drawings using a straightedge and compasses are required and all construction lines must be shown. If the figure to be constructed is a subset of the construction line(s), then no deduction should be made for drawing the line(s), e.g., the altitude of a triangle. Failure to label, when required to do so, should result in the proper deduction. If the description of a construction problem is given in lieu of a drawing, not more than 50 percent of the assigned credit, and usually less, may be allowed. If the construction is on Part I, however, then no partial credit may be given. [Ex-10]

2. If, instead of a direct proposition as called for on the examination paper, the student correctly proves a converse of the proposition, 50 percent of the credit may be granted *provided the proofs are analogous and of relatively equal difficulty.*

3. "Reasoning in a circle" should not necessarily invalidate a whole proof. If credit is to be given in such cases, the amount should be determined by the extent to which the work shown could be used in establishing a logical proof.

4. There is a valid concern as to what geometric principles or theorems may legitimately be used as reasons assigned in the proof of original exercises. It is suggested that in the proof of original exercises, students should be cautioned to restrict themselves to the use of propositions as they appear in the current course guide published by the Department and to the use of those propositions presented by their instructor. [Ex-22; Ex-26]

5. In general, equal credit is not assigned to each step of a geometric proof. Some steps are more important than others and the assignment of credits should reflect this. For example, (2,2,3,2,1) might be the credits assigned to a "statement-reason proof" consisting of 5 steps. It is always beneficial to rate the proof as a whole in determining the credits for the individual steps. [Ex-14(iii),(iv); Ex-22]

6. In the case of original exercises, credit not to exceed 20 percent may be granted for drawing the figure not given and stating the hypothesis and conclusion *in terms of the figure*, provided that it is obvious that the student's work shows an understanding of the problem.

7. In geometry, proofs involving special cases should receive little or no credit. Special cases which do preserve generality should never receive more than 50 percent credit.

8. For geometric verbal problems with a credit allotment of $[a, b]$, allow a credits for the analysis part, which consists of drawing the correct figure, the statement of formula(s) to be used, and the formation of the equation(s) from which the unknown parts are to be obtained. The computational part is worth b credits. [Ex-25(b)]

9. Answers that may be left in terms of π are considered equivalent to those obtained by using $\frac{22}{7}$ or 3.14.

10. The use of the incorrect pre-image when performing a transformation from a point to its image is considered a minor error as long as it is clear what point has been used as the pre-image. The principle in this case is the proper use of the transformation. [Ex-35(c),(d)]

GRAPHING and COORDINATE GEOMETRY

Note–See sections on circular functions and probability and statistics for specific comments about graphing in those areas.

1. Graphs should be drawn with straightedge (ruler) and compasses, where applicable, and with reasonable accuracy.

2. The common deductions for graphing problems are:

(a) Axes not labeled	−1
(b) Failure to label graphs when more than one is to be drawn on the same set of axes (no deduction if only one line or curve is not labeled)	−1 [Ex-33(a)]
(c) Incorrect solid or dotted line	−1 [Ex-33(a)]
(d) Failure to complete graph over the specified domain	−1 or −2 [Ex-23]
(e) Failure to indicate a scale other than the unit scale	−1
(f) Nonlinear equations graphed as a set of line segments	−1

3. When rating the graphing of a system of linear equations with a credit allotment of [8,2], allow:
   (a) 3 credits for each correct line
   (b) 2 credits for listing the coordinates of the correct solution (the point of intersection of the lines drawn)
   (c) 2 credits for the check

4. When the graphing of a pair of inequalities with a credit allotment of [8,2] is rated, each inequality is worth 4 credits with:
   (a) 3 credits for graphing the correct solid or dotted line
   (b) 1 credit for shading the correct region [Ex-33(a)]

5. When rating the graphing of a system of equations consisting of one linear and one quadratic equation with a credit allotment of [8,2], allow:
   (a) 4 credits for the correct graph of the quadratic equation
   (b) 2 credits for the correct line
   (c) 2 credits for listing the coordinates of the correct solution(s) (the point(s) of intersection of the curves drawn)
   (d) 2 credits for the check

6. If a problem calls for the graphical solution of a pair of simultaneous equations, then only a graphical solution using coordinate axes should receive full credit. A correct algebraic solution should receive no more than 50 percent. If required, a proper check should receive the specified credit.

7. When the problem involves finding the area of a polygon whose vertices are given:
   (a) The correct answer with no work shown receives 20 percent credit [Ex-21(i)]
   (b) A diagram which indicates the method of solution, along with the correct solution, should receive full credit [Ex-21(ii)]

8. In coordinate geometry proofs, the mere presence of the appropriate computation is not sufficient for full credit. Conclusions should be stated, with appropriate written reasons supplied. [Ex-17; Ex-26]

9. The coordinates of a point (a,b) may also be expressed as the pair of equations $x = a$ and $y = b$. The expression $\{a,b\}$ or a, b is not an acceptable equivalent for the ordered pair (a,b).

10. Algebraically equivalent equations of lines, circles, etc., are acceptable. [Ex-8(i)-(iii)]

CIRCULAR FUNCTIONS

1. Full credit for the proof of an identity cannot be granted unless it is evident that the procedure followed and the transformations used in the solution are clearly demonstrated by the pupil. For an identity, the rater should deduct 50 percent for the use of cross multiplication. [Ex-11]

2. Numerical illustrations are not to be accepted in lieu of proofs; i.e., in proving $\sin A + \cot A \cdot \cos A = \csc A$, no credit is allowed if $m\angle A = 45$ is used.

3. Verbal problems having to do with the solution of triangles involve much more than mere computation. Credit should be divided between the analysis of the problem and the computation. The analysis should include the drawing of a correct figure, a statement of the formula or formulas to be used, and the formation of the equation or equations from which the unknown parts are to be obtained. A correctly labeled diagram only should receive no more than 20 percent. [Ex-27]

4. The common deductions for trigonometric graphs are:

(a) Axes not labeled	−1
(b) x-axis not labeled in radians	−1 [Ex-18(a)]
(c) Failure to label graphs when more than one is to be drawn on the same set of axes (no deduction if only one line or curve is not labeled)	−1 [Ex-18(a)]
(d) Graph drawn as a set of line segments	−1

5. Trigonometric graphs which are to be *sketched* should be done on appropriate graph paper; however, no deduction should be made for a correct, legible, and clearly labeled graph not done on graph paper.

6. For graphing a trigonometric equation worth a total of 4 credits, the deductions are:

(a) Major flaw with the interval	−2 [Ex-18(a)]
(b) Amplitude and frequency interchanged	−2 [Ex-18(a)]
(c) Failure of graph to pass through such key points as maximum or minimum points and x- and y-intercepts	−1

7. The graph of $y = \sin x$ or $y = \cos x$ in place of a graph with an amplitude and/or frequency other than 1 should receive 1 credit.

8. A correct rational approximation, as obtained from the table provided in the examination booklet (4 decimal places), for irrational values of trigonometric functions is an acceptable equivalent, unless otherwise specified. [Ex-3]

9. Unless the measure of an angle is specifically requested in degrees or radians, either form may be accepted.

LOGIC

1. The credits assigned to a truth table are divided among the columns. Usually a simple negation column receives 1 credit, while a conjunction, implication, etc., should receive 2 credits. Each error in a column is a 1-credit deduction; however, the total deduction for any one column should never exceed the total credits assigned to that column. [Ex-16]

2. A logic proof should demonstrate a logically correct sequence of conclusions obtained by applying the laws of logic to premises and previously drawn conclusions. The form should be consistent with that presented by the instructor (two-column, one-column, narrative). In no case should a mere designation of the truth values of the variables involved be construed as a proof. The symbolizing of the statements given should not receive more than 20 percent. [Ex-14(i)-(iv)]

PROBABILITY and STATISTICS

1. For questions where no partial credit is allowed, no credit should be allowed for answers left in ${}_nC_r$, $\binom{n}{r}$, ${}_nP_r$, or factorial notation; however, if partial credit is allowed, a deduction of 1 credit would apply for each answer left in this notation form. [Ex-12(c)]

2. Histograms should be drawn on graph paper using a straightedge. An accurate, legible, properly labeled histogram, which is not drawn on graph paper, should not receive a deduction.

3. The common deductions for graphs of frequency histograms or cumulative frequency histograms are as follows:

(a) Failure to label the axes	−1 [Ex-19(b)]
(b) Failure to indicate uniform scale	−1
(c) Drawing of a bar graph instead of a histogram	−1

4. Usually histograms are illustrated with an ascending horizontal scale; however, a descending horizontal scale is permissible. The graph drawn must be consistent with the table. Percentiles and quartiles must be consistent with the data given in the original question. [Ex-19(a),(b),(c)]

5. In rating a standard deviation problem, no credit is given for the formula, since it appears on the formula page. The credits are usually distributed among the following:
   (a) Correct calculation of the mean
   (b) Correct sum of the squares of the deviations from the mean
   (c) A correct substitution into the formula
   (d) Finding the square root to the desired accuracy

   In a set of grouped data, failure to use the frequencies is a major error and should result in a deduction of 30 percent. [Ex-13]

Sample Student Responses to Part I Questions

1. *Question*

Evaluate: $\frac{5!}{2!}$

Correct Solution

60

Student Response

$\frac{120}{2}$

(+2)

Rating Explanation

Single fractions need not be in lowest terms unless the directions say to simplify or express in lowest terms. [ALG-1a]

---

2. *Question*

Find antilog 1.4804.

Correct Solution

30.23

Student Response

30.2

(+0)

Rating Explanation

Unless otherwise specified, antilogarithms should be given to four significant figures. [GEN-26]

---

3. *Question*

Find the value of cos (Arc sin $\frac{2}{3}$).

Correct Solution

$\frac{\sqrt{5}}{3}$

Student Response

.7454

(+2)

Rating Explanation

A rational approximation of a correct answer obtained by interpolation from tables [CIR FN-8; GEN-26]

---

4. *Question*

Find the negative root of the equation $|3x - 2| + 4$.

Correct Solution

$-\frac{2}{3}$

Student Response

$\{2, -\frac{2}{3}\}$

(+0)

Rating Explanation

Only the negative root was asked for and student does not differentiate between positive and negative root. [ALG-3b]

---

5. *Question*

Solve for x:

$x + y = 12$

$y = 3x$

Correct Solution

3

Student Responses

(i) (3,9) or $x = 3, y = 9$ (+2)

(ii) (3,6) or $x = 3, y = 6$ (+0)

Rating Explanations

(i) Full credit when both answers are correct [ALG-3a]

(ii) Second value is not correct. [ALG-3a]

6. *Question*

Find the slope of the line whose equation is $y = \frac{-2x}{3} + 2$.

Correct Solution

$-\frac{2}{3}$

Student Responses

(i) slope = $-\frac{2}{3}$ (+2)
y-intercept = 2

(ii) slope = $-\frac{2}{3}$ (+0)
y-intercept = 3

(iii) $m = -\frac{2}{3}$ (+0)
$b = 2$

(iv) $m = -\frac{2}{3}$ (+2)

Rating Explanations

(i) Both are correct and clearly identified. [ALG-3c]

(ii) y-intercept not correct [ALG-3c]

(iii) Slope not clearly identified [ALG-3c]

(iv) Acceptable answer

---

7. *Question*

Find the positive square root of 49.

Correct Solution

7

Student Response

± 7 (+0)

Rating Explanation

Only the positive root was asked for. [ALG-3b]

---

8. *Question*

Write an equation of the line whose slope is $\frac{2}{3}$ and which passes through the point (6, – 1).

Correct Solution

$y = \frac{2}{3}x - 5$

Student Responses

(i) $y + 1 = \frac{2}{3}(x-6)$ (+2)

(ii) $2x - 3y = 15$ (+2)

(iii) $3y + 3 = 2x - 12$ (+2)

Rating Explanations

(i) An equivalent equation [GRAPH-10]

(ii) Same as (i) [GRAPH-10]

(iii) Same as (i) [GRAPH-10]

---

9. *Question*

If $a * b$ is a binary operation defined as $a + 2b$, evaluate $5 * 3$.

Correct Solution

11

Student Response

$5 + 2(3)$ (+0)

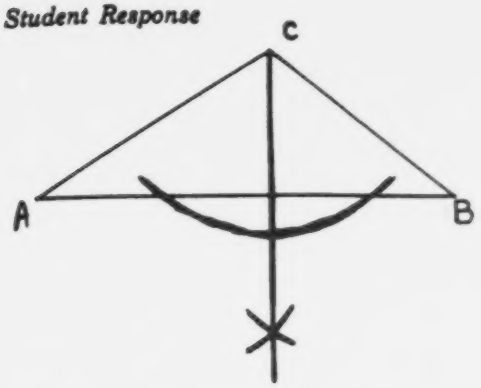
Rating Explanation

Result depends upon the proper use of the order of operations. [ALG-1e]

10. *Question*

On the answer sheet construct the altitude to side $\overline{AB}$ of $\triangle ABC$. [2]

Student Response



(+2)

Rating Explanation

The altitude required is a subset of the line necessary for the construction. [GEOM-1]

Sample Student Responses to Part II or Part III Questions

11. *Question*

For all values of θ for which the expressions are defined, show that the following is an identity:

$$\frac{\sin 2\theta}{\tan \theta} = \frac{2}{1 + \tan^2 \theta}$$ [5]

Student Response

$$\frac{\sin 2\theta}{\tan \theta} = \frac{2}{1 + \tan^2 \theta}$$

$$\sin 2\theta (1 + \tan^2 \theta) = 2 \tan \theta$$

$$2 \sin \theta \cos \theta (\sec^2 \theta) = \frac{2 \sin \theta}{\cos \theta}$$

$$\frac{2 \sin \theta}{\cos \theta} = \frac{2 \sin \theta}{\cos \theta}$$

(+1 credit)

Rating Explanation

Cross multiplication results in a 50 percent deduction. (-3) One credit is deducted for work not shown. In Course III, formulas are given; therefore, credit is only allowed for substitution. [CIR FN-1]

12. *Question*

A florist has 5 red roses, 3 pink roses, and 2 yellow roses.

a How many different bouquets can be made using 3 roses? [2]

b How many of these bouquets can be formed using 3 red roses? [2]

c How many of these bouquets can be formed using 2 pink roses and 1 yellow rose? [2]

d What is the probability that one of these bouquets consists of 3 red roses? [2]

e What is the probability that one of these bouquets consists of 2 pink roses and 1 yellow rose? [2]

Correct Solution

a 120

b 10

c 6

d $\frac{1}{12}$

e $\frac{1}{20}$

Student Response

a ${}_{10}C_3 = \frac{10!}{3!7!} = 120$ (+2)

a correct

b ${}_5C_3 = \frac{5!}{3!2!} = 10$ (+2)

b correct

c ${}_3C_2 \cdot {}_2C_1$ (+1)

c did not simplify [PROB-STAT-1]

d $\frac{10}{120} = \frac{1}{12} = 12\%$ (+1)

d additional incorrect information given [GEN-23]

e $\frac{{}_3C_2 \cdot {}_2C_1}{120}$ (+2)

e correct, based on answers in *a* and *c* [GEN-12]

(+8 credits)

Rating Explanation

13. *Question*

Find, to the *nearest tenth*, the standard deviation for the following data. [8]

mean (x_i)	frequency (f_i)
20	3
22	2
23	2
25	2
30	1

Correct Solution

2.9

Student Response

x_i f_i $x_i - \bar{x}$ $(x_i - \bar{x})^2$

20	3	-4	16
22	2	-2	4
23	2	-1	1
25	2	1	1
30	1	6	36

$\frac{120}{5} = 24$

$\frac{58}{5} = 11.4$

S.D. $= \sqrt{11.4} = 3.38$

$\frac{1.0569}{2} = .5284$

(+4 credits)

Rating Explanation

One point is deducted for a division error in the last column and one point for not rounding. Also, 2 credits (30 percent of 8) are deducted for failure to use the frequencies. [GEN-14,19; PROB-STAT-5] Credit may be granted for the square root calculation since the work is shown. [GEN-13]

14. *Question*

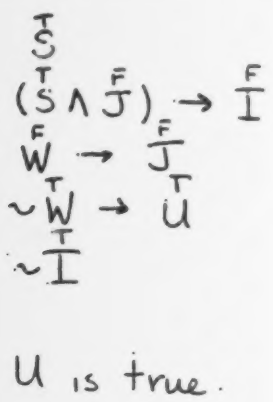
Given: Jason is smart.
If Jason is smart and has a job, then he will earn a good income.
If Jason works hard, then he will have a job.
If Jason does not work hard, then he will be unhappy.
Jason does not have a good income.

Let *S* represent: "Jason is smart."
Let *W* represent: "Jason works hard."
Let *J* represent: "He has a job."
Let *I* represent: "He will earn a good income."
Let *U* represent: "He will be unhappy."

Prove: Jason will be unhappy. [10]

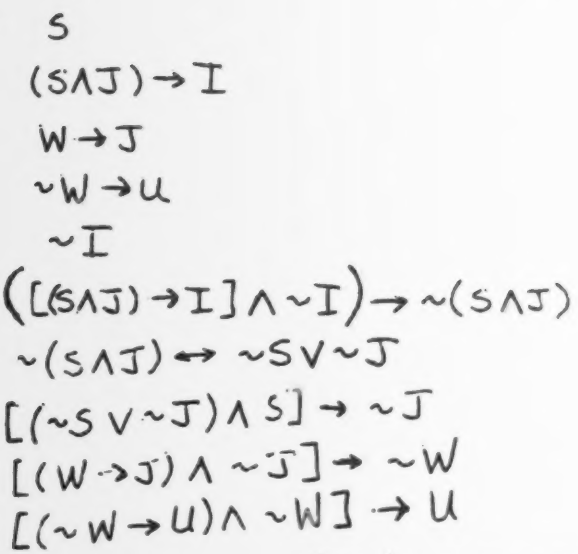
Student Responses

(i)



(+2 credits)

(ii)



(+10 credits)

Rating Explanations

(i) Mere designation of the truth values is not a proof. Symbolizing only receives a maximum of 20 percent of the credit. [LOGIC-2]

(ii) Full credit may be allowed if this is the method used by the instructor. [LOGIC-2]

(iii)

S

$(S \wedge J) \rightarrow I$

$W \rightarrow \sim J$

$\sim W \rightarrow U$

$\sim I$

$(S \wedge J) \rightarrow I$

$\sim I$ (+2)

$\therefore \sim S \vee \sim J$

S (+2)(-1)

$\therefore \sim J$ (+2)

$W \rightarrow \sim J$

$\therefore \sim W$ (+1)

$\sim W \rightarrow U$ (+2)

$\therefore U$ (+9 credits)

(iii) **If the form is consistent with the method used by the instructor, then the only deduction here results from not showing the transition from $\sim(S \wedge J)$ to $\sim S \vee \sim J$. [LOGIC-2; GEN-21; GEOM-5]**

(iv)

Statements	Reasons
1. S	1. Given
2. $(S \wedge J) \rightarrow I$	2. "
3 $W \rightarrow \sim J$	3. "
4 $\sim W \rightarrow U$	4. "
5. $\sim I$	5 "
6. $\sim(S \wedge J)$	6. law of contrapositive inference (2,5)
7. $\sim S \vee \sim J$	7. DeMorgan's laws (6)
8. J	8. same as 6 (1, 7)
9. W	9. law of converse (3, 8)
10. U	10. law of inverse (4, 9)

(+5 credits)

(iv)

steps 1-5	correct symbolizing of arguments (+2)
step 6	If the form is consistent with the method used by the instructor, no deduction should be made. (+2)
step 7	same as 6 (+1)
steps 8-10	All are invalid conclusions. (+0) [LOGIC-2, GEOM-5]

15. *Question*

Using the accompanying table, solve for all x:

$5 * x = 5$ [2]

*	1	3	5	7
1	1	1	1	1
3	1	3	3	3
5	1	3	5	5
7	1	3	5	7

Correct Solution

{5,7}

Student Responses

(i) {5} (+1)

(ii) {7} (+1)

(iii) {1, 5} (+1)

(iv) {1, 5, 7} (+1)

(v) {1, 3, 5, 7} (+0)

(vi) {1, 3, 5} (+0)

Rating Explanations

(i) 1 credit deducted for omitted answe.

(ii) same as (i)

(iii) 1 credit deducted for an incorrect answe

(iv) same as (iii)

(v) 2 credits deducted for incorrect answers

(vi) same as (v)

[ALL ALG-13]

---

16. *Question*

On your answer paper, construct a truth table for the statement $\sim(p \rightarrow \sim q) \rightarrow (p \wedge q)$. [10]

Student Response

p	q	~q	p∧q	p→~q	~(p→~q)↔(p∧q)
T	T	F	T	F	T
T	F	T	T	T	F
F	T	F	T	T	F
F	F	T	F	T	F

(+1)(+1)(+1)(+0) (+2) (+1)

(+6 credits)

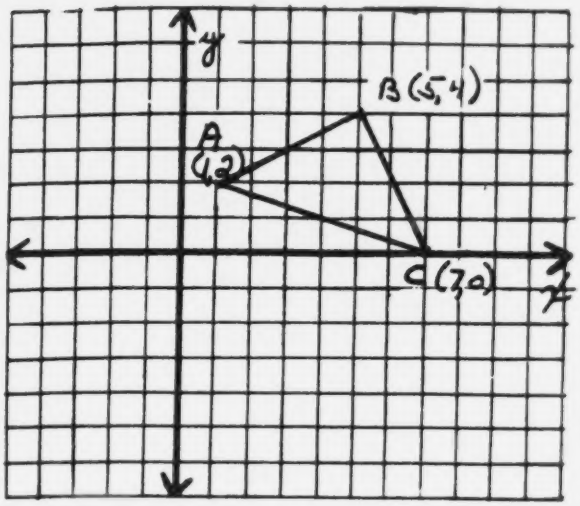
Rating Explanations

Fourth column has two mistakes. (-2) [LOGIC-1]
$\sim(p \rightarrow \sim q)$ omitted (-1) [LOGIC-1]
First three entries of last column are correct based on incorrect earlier responses. [GEN-12]
Last entry of last column is wrong based on earlier responses. (-1) [GEN-12]

17. *Question*

The vertices of triangle ABC are $A(1,2)$, $B(5,4)$, and $C(7,0)$. Prove that triangle ABC is an isosceles triangle. [10]

Student Response



$AB = \sqrt{16+4} = \sqrt{20}$

$BC = \sqrt{4+16} = \sqrt{20}$

$AC = \sqrt{36+4} = \sqrt{40}$ (+3)

$(\sqrt{20})^2 + (\sqrt{20})^2 = (\sqrt{40})^2$ (+3)

$20 + 20 = 40$ (+6 credits)

Rating Explanation

Conclusions with appropriate reasons are not given. [GRAPH-8]

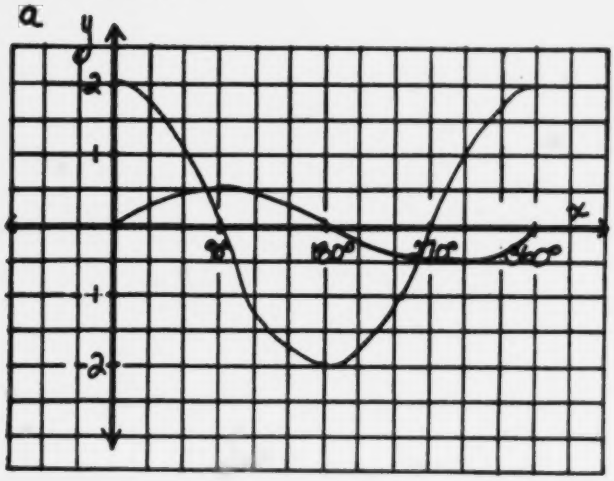
18. *Question*

a On the same set of axes, sketch and label the graphs of $y = \sin \frac{1}{2}x$ and $y = 2 \cos x$ as x varies from $-\pi$ to π radians. [8]

b From the graphs made in part *a*, how many values of x in the interval $-\pi \leq x \leq \pi$ satisfy the equation $\sin \frac{1}{2}x = 2 \cos x$? [2]

Student Response

a



b 2

(+ 4 credits)

Correct Solution

b 2

Rating Explanations

The graph of $y = \sin \frac{1}{2}x$ has the amplitude and frequency interchanged. (-2) [CIR FN-6b]

The *x*-axis is labeled in degrees, not radians. (-1) [CIR FN-4b]

Curves are not labeled. (-1) [CIR FN-4c]

Curves are not drawn over the proper interval. (-2) [CIR FN-6a]

No credit is deducted on part *b*, since answer is correct based on the curves drawn in part *a*. [GEN-12]

19. *Question*

The table below shows the distribution of bowling scores.

Interval	Frequency	Cumulative Frequency
91 – 100	10	
111 – 130	13	
131 – 150	8	
151 – 170	4	
171 – 190	4	
191 – 210	5	

a On your answer paper, copy the table and complete the Cumulative Frequency column. [2]

b Using the data in the Cumulative Frequency column of the table, draw a cumulative frequency histogram. [4]

c Use your results from part *b* to answer the following questions.

(1) In which interval does the median lie? [2]

(2) In which interval does the upper quartile lie? [2]

Correct Solution

c (1) 111 – 130

(2) 151 – 170

Student Response

a

Interval	Frequency	Cumulative Frequency
91-110	10	44
111-130	13	34
131-150	8	21
151-170	4	13
171-190	4	9
191-210	5	5

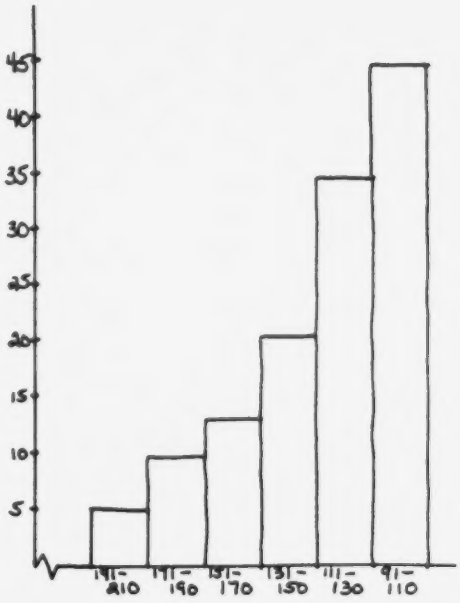
(+ 2)

c (1) 111-130 (+ 2)

(2) 111-130 (+ 0)

(+ 7 credits)

b



(+ 3)

Rating Explanation

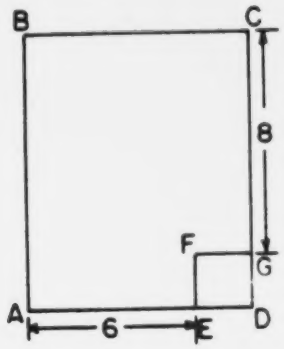
a Addition is correct for the cumulative frequency data. (+ 2) [PROB-STAT-4]

b The graph was not labeled; otherwise it is correct based on part *a*. (–1) [PROB-STAT-3a; PROB-STAT-4]

c Answer (2) is incorrect, based on the given data. (–2) [GEN-12; PROB-STAT-4]

20. *Question*

In the accompanying diagram, *ABCD* is a rectangle and *DEFG* is a square. The area of *ABCD* is 80, *CG* = 8, and *AE* = 6. Find the length of the side of square *DEFG*. [*Only an algebraic solution will be accepted.*] [5,5]



Correct Solution

2

Student Response

Let x = the length of the side of the square

7·9 = 63

8·10 = 80 ✓

6 + x = 8 6 + 2 = 8

8 + x = 10 8 + 2 = 10

So x = 2 and the side of the square is 2.

(+3 credits)

Rating Explanation

One point is given for the analysis (representing the unknown) and 2 points for the answer. [ALG-5,6,9; GEN-15]

21. *Question*

Find the area of pentagon $ABCDE$ whose vertices are $A(-2,-5)$, $B(-2,2)$, $C(2,4)$, $D(5,2)$, and $E(4,-2)$. [10]

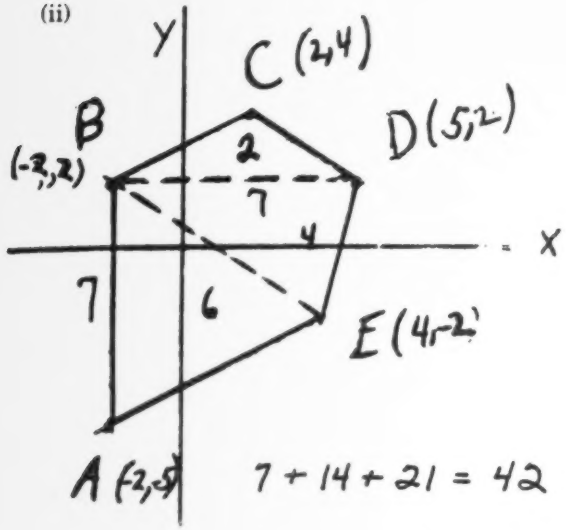
Correct Answer

42

Student Responses

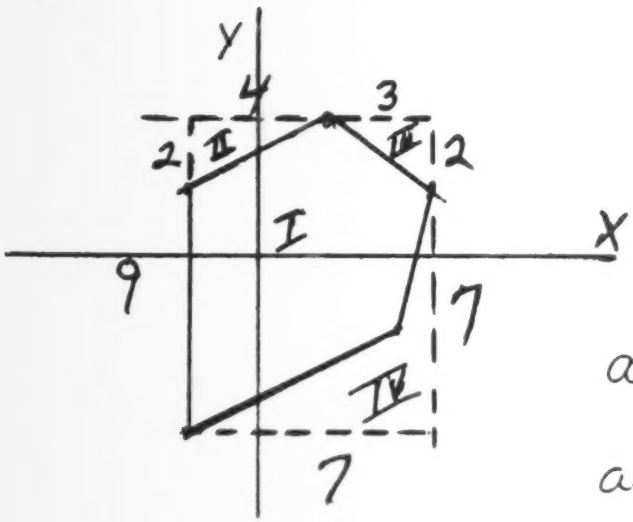
(i) 42 (+2)

(ii)



(+10 credits)

(iii)



area I = area of rectangle − (area II + area III + area IV)
area I = 63 − (4 + 3 + 24½)
= 63 − 31½
= 31½

(+6 credits)

Rating Explanations

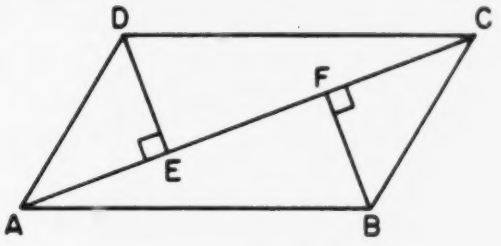
(i) answer only [GRAPH-7a]

(ii) Since the method of solution can be clearly determined from the graph and since mental arithmetic and a simple formula can be used, full credit is granted. [GRAPH-7b, GEN-13,21]

(iii) Improper analysis, since region IV is not a triangle (-2) and the resulting calculation is not as difficult as the correct solution. (-2) [GEN-14,18]

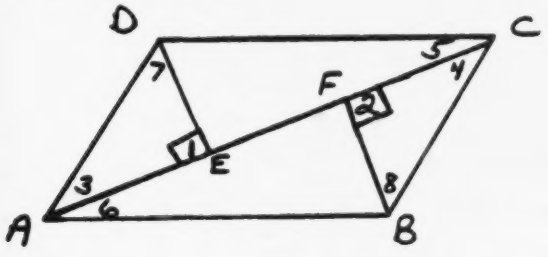
22. ***Question***

Given: quadrilateral $ABCD$, diagonal $\overline{AEFC}$, $\overline{DE} \perp \overline{AC}$, $\overline{BF} \perp \overline{AC}$, $\overline{AE} \cong \overline{CF}$, and $\overline{DE} \cong \overline{BF}$.



Prove: $ABCD$ is a parallelogram. [10]

Student Response



Prove: ABCD is a ▱

Statements	Reasons
1. diagonal $\overline{AEFC}$, $\overline{DE} \perp \overline{AC}$, $\overline{BF} \perp \overline{AC}$, $\overline{AE} \cong \overline{CF}$, $\overline{DE} \cong \overline{BF}$	1. Given
2. $\angle 1 \cong \angle 2$	2. All rt. angles are $\cong$
3. $\triangle ADE \cong CBF$	3. SAS
4. $\angle 3 \cong \angle 4$, $\angle 7 \cong \angle 8$, $\overline{AD} \cong \overline{CB}$	4. CPCTC
5. $\overline{AC} \cong \overline{CA}$	5. Identity
6. $\triangle DAC \cong \triangle BCA$	6. SAS
7. $\angle 5 \cong \angle 6$, $\overline{DC} \cong \overline{BA}$	7. CPCTC
8. $\overline{DC} \parallel \overline{AB}$, $\overline{AD} \parallel \overline{BC}$	8. $\cong$ corresponding angles make $\parallel$ lines
9. ABCD is a ▱	9. If both pairs of opposite sides of a quad. are $\cong$, the quad. is a ▱.

Rating Explanation

Assuming that the form of the reasons is consistent with that presented by the instructor: [GEOM-4]

$\angle 1$ and $\angle 2$ are not identified as right angles. (–1) [GEOM-5; GEN-21]

In step 4, $\angle 7 \cong \angle 8$ is irrelevant correct information which does not detract from the nature of the proof. [GEN-23]

Step 8 contains irrelevant incorrect information (the reason). (–1) [GEN-23]

(+8 credits)

23. *Question*

Given the equation: $y = -x^2 + 4x - 3$

Draw the graph of the equation using all values of x such that $-2 \leq x \leq 6$. [6]

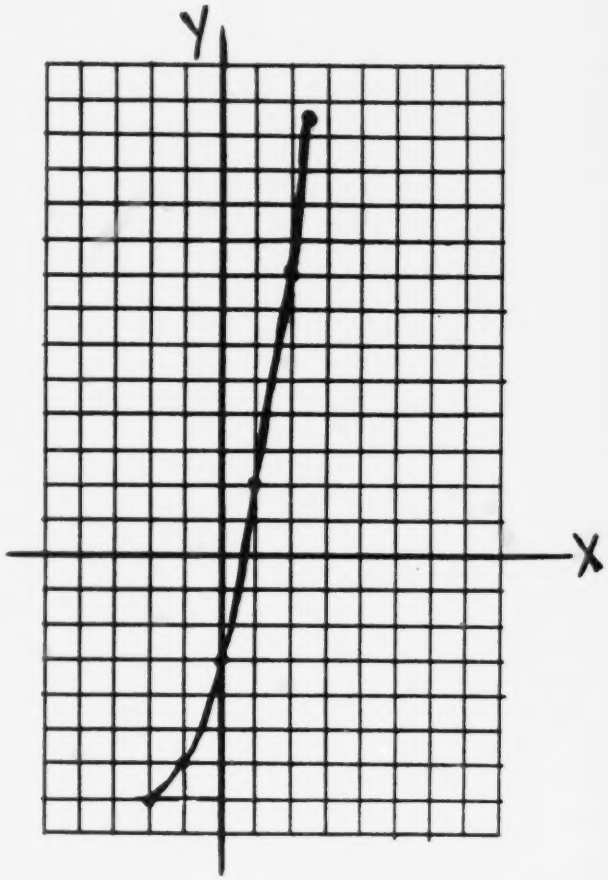
Student Response

$y = -x^2 + 4x - 3$

x	y
-2	$4 - 8 - 3 = -7$
-1	$1 - 4 - 3 = -6$
0	$-3 = -3$
1	$1 + 4 - 3 = 2$
2	$4 + 8 - 3 = 9$
3	$9 + 12 - 3 = 18$
4	$16 + 16 - 3 = 29$

Rating Explanation

Improper evaluation of $-x^2$ is a major error in principle. (-3) [GEN-14]
Failure to graph over the entire interval (-1) [GRAPH-2d]



(+2 credits)

24. *Question*

Mary chose one of the four numbers, 1, 2, 3, and 6, at random. She then chose one of the two numbers, 1 and 5, at random.

a Draw a tree diagram or list the sample space of all possible pairs of numbers that Mary could choose. [3]

b Find the probability that Mary chose an even number first followed by an odd number. [2]

c Find the probability that Mary chose *at least* one even number. [2]

d Find the probability that both choices were the same number. [2]

e Find the probability that Mary chose two even numbers. [1]

Correct Solution

b $\frac{4}{8}$

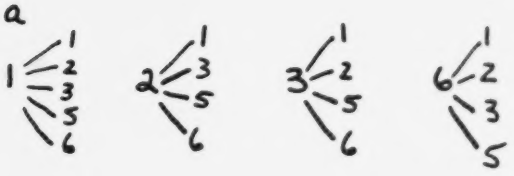
c $\frac{4}{8}$

d $\frac{1}{8}$

e 0

Student Response

a



(+1)

b $\frac{6}{17}$ (+2)

c $\frac{12}{17}$ (+2)

d $\frac{2}{17}$ (+0)

e 0 (+0)

(+5 credits)

Rating Explanation

a An understanding of the use of tree diagrams for two choices is shown; however, second choice options are incorrect. [GEN-14]

b,c Correct for the information in part *a* [GEN-12]

d,e Incorrect for diagram used in part *a* [GEN-12]

25. ***Question***

In triangle ABC, $m\angle C = 90$, $AC = x$, $BC = x + 3$, and $AB = x + 5$.

a Write an equation in terms of x which can be used to find AC. [3]

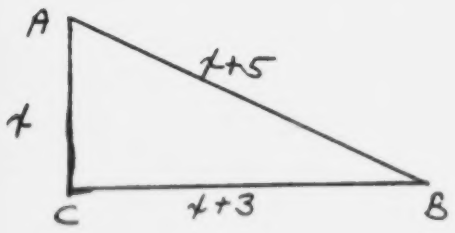
b Find AC. [*Answer may be left in radical form.*] [7]

Correct Solution

a $x^2 + (x + 3)^2 = (x + 5)^2$

b $2 + 2\sqrt{5}$ or $\frac{4 + \sqrt{80}}{2}$

Student Response



a $2x^2+6x+9=x^2+10x+25$

b $x^2-4x+16=0$

$$x=\frac{4\pm\sqrt{16-4(1)(16)}}{2}$$

Rating Explanation

Part *a* has an answer which is algebraically equivalent to the given answer. (+3) [GEN-10]

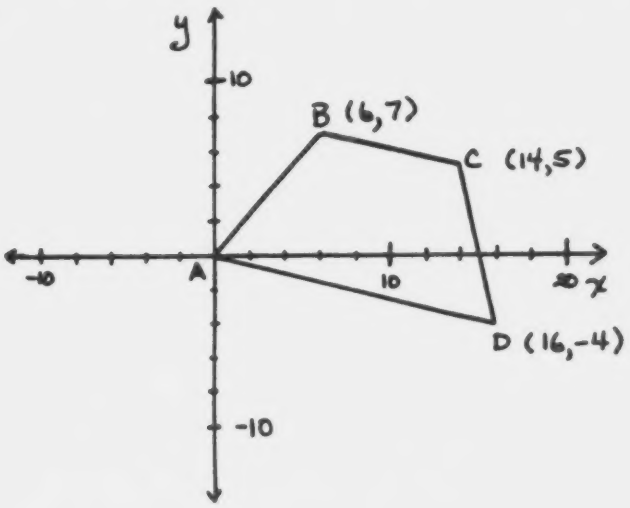
Correct squaring and combining of similar terms (+2) [GEOM-8]
Computational error (+16 instead of −16) (−1) [GEN-14]
Correct substitution into formula (+2) [GEOM-8]
Simplification and elimination of extraneous roots omitted. (−2) [ALG-10]

(+7 credits)

26. *Question*

The vertices of a quadrilateral are $A(0,0)$, $B(6,7)$, $C(14,5)$, and $D(16,-4)$. Prove that $ABCD$ is an isosceles trapezoid. [10]

Student Response



$m_{\overline{BC}} = \frac{5-7}{14-6} = \frac{-2}{8} = -\frac{1}{4}$

$m_{\overline{AD}} = \frac{-4}{16} = -\frac{1}{4}$

$\therefore \overline{AD} \parallel \overline{BC}$ since lines with equal slopes are $\parallel$

$\therefore$ ABCD is a trapezoid because it has a pair of $\parallel$ sides

$AB = \sqrt{36+49} = \sqrt{85}$

$CD = \sqrt{(16-14)^2+(-4-5)^2} = \sqrt{2+81} = \sqrt{83}$

AB and CD aren't equal so ABCD is not isosceles.

(+6 credits)

Rating Explanation

Slopes of $\overline{BC}$ and $\overline{AD}$ are correct and reason is given for the lines being parallel. (+3) [GRAPH-8]
It has not been shown that $\overline{AB}$ is not parallel to $\overline{CD}$. (-3) [GEOM-4; GEN-14]
Computation error in calculating the length of $\overline{CD}$ (-1) [GEN-14]
The correct calculation of the length of $\overline{AB}$, together with an appropriate conclusion drawn, based on these calculations (+3) [GEN-12]

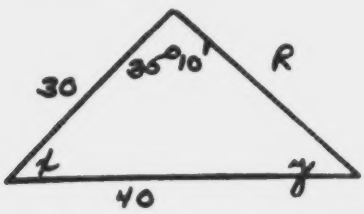
27. *Question*

Two forces of 30 pounds and 40 pounds act upon a body, forming an acute angle with each other. The angle between the resultant and the 30-pound force is 35° 10′. Find, to the *nearest ten minutes*, the angle between the two given forces. [10]

Correct Solution

60° 50′

Student Response



$$\frac{40}{\sin 35°10'} = \frac{30}{\sin y}$$

$$40 \sin y = 30(\sin 35°10')$$

$$\sin y = \frac{30(\sin 35°10')}{40} = \frac{3(.1440)}{4} = .4320$$

$$y = 64°30'$$

$$x = 180 - (64°30' + 35°10')$$

$$x = 180 - (99°40')$$

$$x = 80°20'$$

(+6 credits)

Rating Explanation

Major error in principle (+1) (-3) [GEN-14; CIR FN-3]
Correct evaluation of sin y (+3)
Improper interpretation of table values (-1) [GEN-14]
Correct value of x based on improper analysis (+2) [GEN-12]

28. *Question*

Find to the *nearest degree*, all values of θ in the interval $0° \leq \theta \leq 360°$ which satisfy the equation $3 \cos 2\theta + \sin \theta - 2 = 0$. [10]

Correct Solution

30°, 150°, 199°, 341°

Student Response

$3\cos 2\theta + \sin\theta - 2 = 0$

$3(1-2\sin^2\theta)+\sin\theta-2=0$

$-6\sin^2\theta + \sin\theta + 1 = 0$

$6\sin^2\theta - \sin\theta - 1 = 0$

$(3\sin\theta-1)(2\sin\theta+1) = 0$

$\sin\theta = \frac{1}{3}$ $\quad$ $\sin\theta = -\frac{1}{2}$

$20°, 160°$ $\quad$ $210°, 330°$

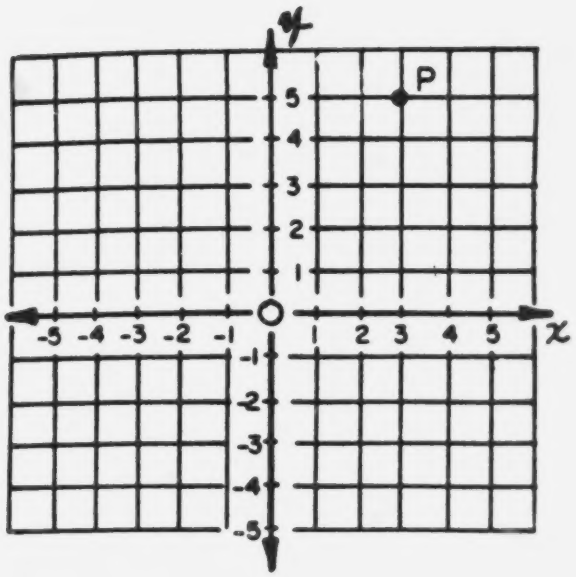
Rating Explanation

Substitution and setting equal to zero (+3)
Factored incorrectly (-2) [GEN-14]
Correct values of sin θ based on factors used (+2) [GEN-12]
Incorrect values from the table (-1) [GEN-14]
Correct values based on interpretation from table (+2) [GEN-12]
Credit for solution is granted because the difficulty level is equivalent to that of the correct solution. [GEN-18]

(+7 credits)

29. *Question*

In the accompanying graph, P is a point whose coordinates are (3,5).



a Describe fully the locus of points at a distance d units from P. [3]

b Describe fully the locus of points at a distance of one unit from the y-axis. [4]

c How many points satisfy the conditions in parts a and b simultaneously for the following values of d?
(1) $d = 2$ [1]
(2) $d = 4$ [1]
(3) $d = 5$ [1]

Correct Solution

a A circle with center P and a radius d

b Two vertical lines, each one unit from the y-axis

c (1) 1
(2) 3
(3) 4

Student Response

a a circle

b the line $x = 1$

c (1) 1
(2) 2
(3) 2

(+6 credits)

Rating Explanation

a Locus is not fully described. (+1) (-2) [GEN-16]

b Same as a (+2) (-2) [GEN-16]

c The rest is correct based on the answers in parts a and b. (+3) [GEN-12]

30. *Question*

Using logarithms, find $\sqrt[3]{0.692}$ to the *nearest hundredth.* [5]

Correct Solution

.88

Student Response

$\log \sqrt[3]{0.692}$

$= \frac{1}{3} \log .692$

$= \frac{1}{3}(.8338-1)$

$= \frac{1}{3}(2.8338-3)$

$= .9446-1$

$= .94$

(+3 credits)

Rating Explanation

Correct theory (40 percent) (+2) [GEN-25]
Incorrect table value (-1) [GEN-14]
Correct division (+1)
Table value not looked up (-1) [GEN-14]

---

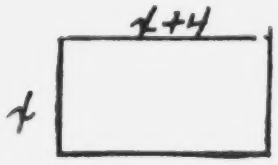
31. *Question*

The length of a rectangle is 4 more than its width. If the area is 32, find the dimensions of the rectangle. [5,5]

Correct Solution

4,8

Student Response



$A = 2w+2l$

$32 = 2(x)+2(x+4)$

$32 = 2x+2x+8$

$32 = 4x+8$

$24 = 4x$

$6 = x$

$10 = x+4$

(+4 credits)

Rating Explanation

Variables are represented in the diagram. (+1) [ALG-6]
Use of the wrong formula is a major error. (-3) [GEN-14]
Correct substitution into the formula (+1)
Solving a much easier equation (+1) (-3) [GEN-18]
Correct second dimension is found. (+1)

32. ***Question***

Solve and check.

$y = 2x$
$y = 2x^2 + 3x - 1$ [8,2]

Student Response

$2x = 2x^2 + 3x - 1$

$2x^2 + x - 1 = 0$

$(2x + 1)(x - 1) = 0$

$2x + 1 = 0 \quad x = 1$

$x = -\frac{1}{2} \quad y = 2(1) = 2$

$y = 2(-\frac{1}{2}) = -1$

$-1 = 2(-\frac{1}{2}) \quad 2 = 2(1)$

(+6 credits)

Correct Solution

$(\frac{1}{2}, 1)$ and $(-1, -2)$

or

$x = \frac{1}{2} \quad x = -1$

and

$y = 1 \quad y = -2$

Rating Explanation

Correct substitution (+1)
Equation is correctly set equal to zero. (+1)
Incorrect factoring (-2)
Solved the incorrect equations correctly for x and y (+4) [GEN-12]
Check only involves substituting into the equation from which the second unknown was determined. (-2) [ALG-12]

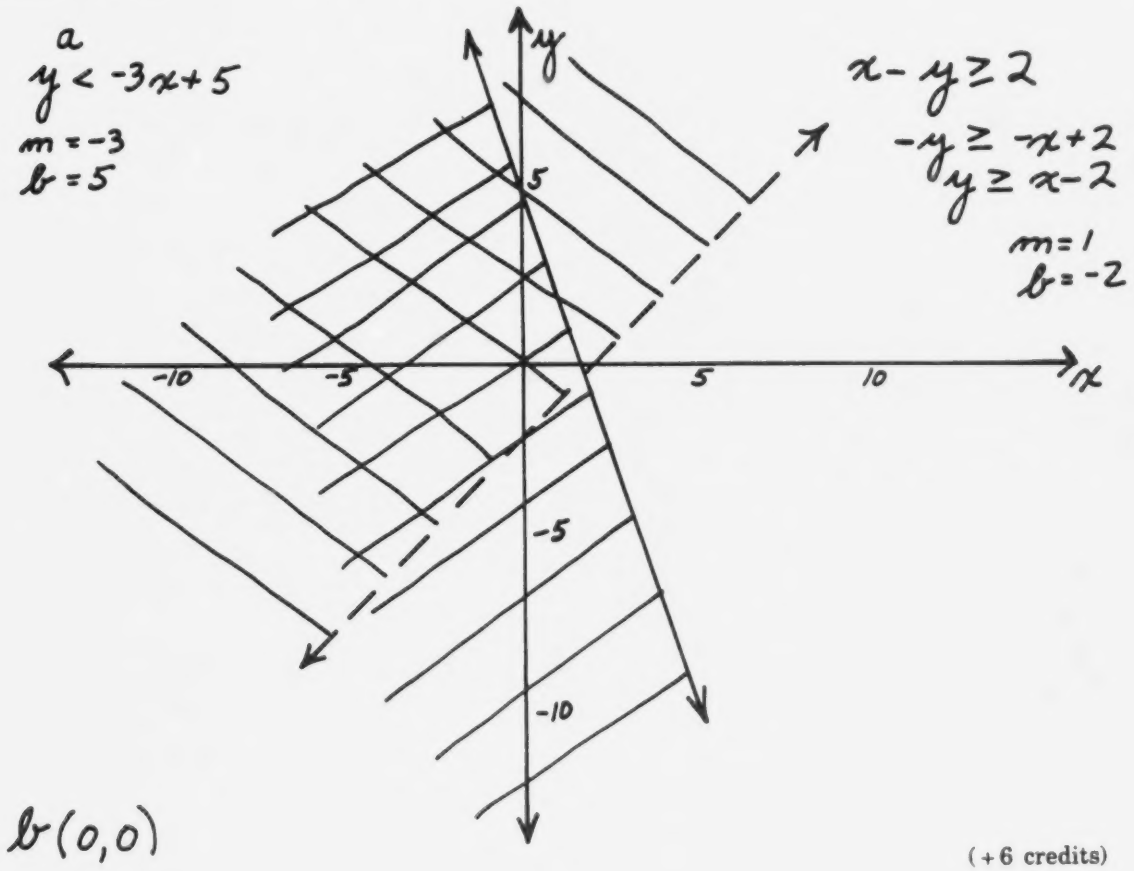
33. *Question*

a On the same set of coordinate axes, graph the following system of inequalities:

$y < -3x + 5$ [8]
$x - y \geq 2$

b Write the coordinates of a point in the solution set of the graph drawn in answer to part *a*. [2]

Student Response



(+6 credits)

Rating Explanation

The student has (due to an algebraic error) shaded the wrong side of the line for the second inequality. (-1) [GRAPH-4b]
The lines are not labeled. (-1) [GRAPH-2b]
The dotted line should be solid. (-1) [GRAPH-2c]
The solid line should be dotted. (-1) [GRAPH-2c]
Part *b* is done correctly for the graph shown. [GEN-12]

34. *Question*

Find the three largest consecutive integers whose sum is less than 86. [*Only an algebraic solution will be accepted.*] [5,5]

Correct Solution

27, 28, 29

Student Responses

(i)

Let x = 1st

$x+1$ = 2nd

$x+2$ = 3rd

$x+x+1+x+2 = 86$

$3x+3 = 86$

$3x = 83$

$x = 27\frac{2}{3}$

#'s are 27, 28, 29

(+10 credits)

Rating Explanations

(i) Writing of an equation instead of an inequality is not an error, since the student has demonstrated a correct interpretation of the problem. [ALG-5,8]

(ii)

Let x = 1st

$x+1$ = 2nd

$x+2$ = 3rd

$x+x+1+x+2 = 86$

$3x+3 = 86$

$3x = 83$

$x = 27\frac{2}{3}$

$x+1 = 28\frac{2}{3}$

$x+2 = 29\frac{2}{3}$

(+6 credits)

(ii) Answers given do not demonstrate a correct interpretation of the problem. (–4) [ALG-5,8]

35. *Question*

The coordinates of the vertices of $\triangle ABC$ are $A(1,6)$, $B(2,9)$, and $C(7,10)$.

a Graph and label $\triangle ABC$. [1]

b Graph and state the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ after a reflection over the line $y = x$. [3]

c Graph and state the coordinates of $\triangle A''B''C''$, the image of $\triangle A'B'C'$ after a reflection over the x-axis. [3]

d Graph and state the coordinates of $\triangle A'''B'''C'''$, the image of $\triangle A''B''C''$ after the transformation $(x,y) \rightarrow (x-5, y+3)$. [3]

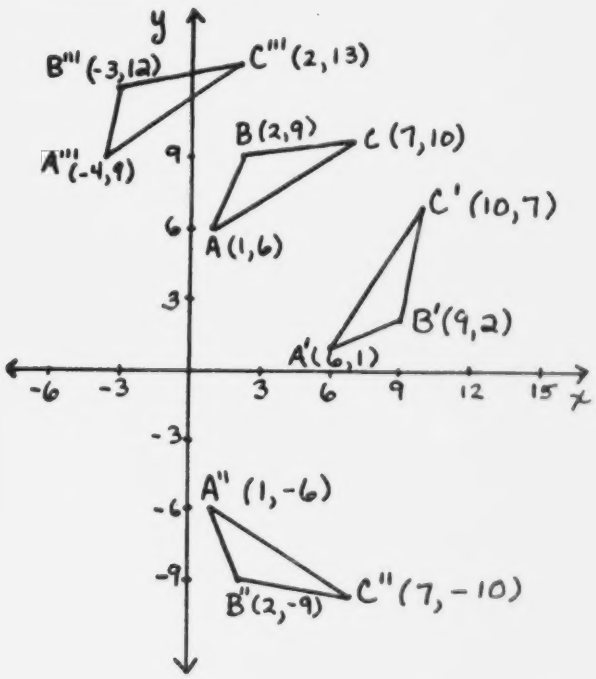
Correct Solution

b $A'(6,1), B'(9,2), C'(10,7)$

c $A''(6,-1), B''(9,-2), C''(10,-7)$

d $A'''(1,2), B'''(4,1), C'''(5,-4)$

Student Response



(+9 credits)

Rating Explanation

a Correct (+1)

b Correct (+3)

c Use of $\triangle ABC$ as pre-image, instead of $\triangle A'B'C'$, is not a cardinal error in principle. (+2)(-1) [GEOM-10]

d $\triangle ABC$ was again used as the pre-image; however a deduction was already made in part c, and the student should not be penalized twice for the same mistake. (+3) [GEOM-10; GEN-12]

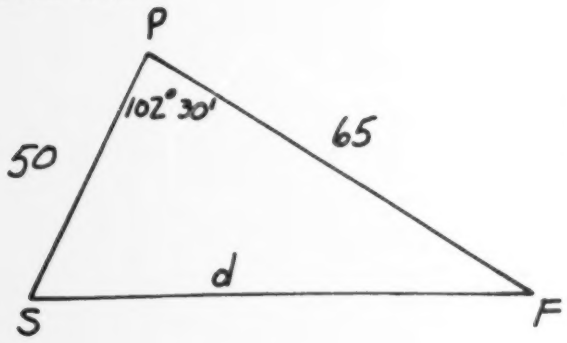
36. *Question*

Sailboat S is 50 meters from a lighthouse located at point P. Fishing boat F is 65 meters from the same lighthouse. If the measure of $\angle SPF$ is 102° 30′, find to the *nearest meter*, the distance between the two boats. [10]

Correct Solution

90

Student Response



$d^2 = 50^2 + 65^2 - 2(50)(65)\cos 102°30'$

$d^2 = 2500 + 4225 - 6500(-\cos 77°30')$

$= 6725 + 6500(.2164)$

$= 6725 + 1407$

$= 8032$

$d = 90$

(+7 credits)

Rating Explanation

Computation of the product (6500) (.2164) is not shown. (-1)
Sum is incorrect. (-1)
Computation of the square root is not shown. (-1) [GEN-13,19]

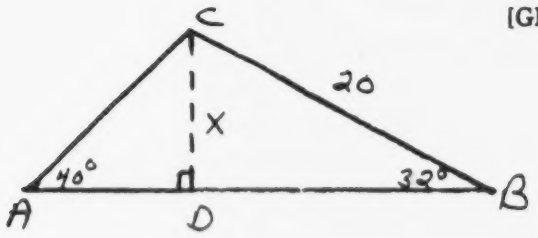
37. ***Question***

In $\triangle ABC$, $m\angle A = 40$, $m\angle B = 32$, and $BC = 20$ meters. Find the length of $\overline{AC}$ to the *nearest meter.* [7]

Correct Solution

16

Student Response



$\frac{X}{20} = \sin 32°$

$X = 20 \sin 32°$

$= 20(.5299)$

$= 10.6$

$\frac{10.6}{AC} = \sin 40°$

$AC = \frac{10.6}{\sin 40°} = \frac{10.6}{.6428}$

$AC = 16.49 = 16$

.5299
20
10.5980

16.49
.6428] 10.6000.00
6428
41720
38568
31520
25712
58080
57852

Rating Explanation

This problem could be solved more directly by using the law of sines; however, the work is correct and mathematically sound, so full credit is granted. [GEN-19,22]

(+7 credits)

Using the same notation, all participants are to add the questions from the August 1985 and June 1986 Sequential Mathematics Course III Regents examinations to the list below.

COURSE III

REGENTS REVIEW MQ6 - PART I QUESTIONS

1. RADIAN MEASURE

186 3; 185 1; 884 2; 684 5; 184 1; 683 29;
183 1, 6; 682 1; 682 8

2. TRANSFORMATIONS/SYMMETRY

186 1, 14, 19, 27, 34; 685 1, 6, 10, 19; 185 2, 6, 18, 22, 34;
684 1, 18, 20, 28; 184 13, 19, 33, 35; 6/83 20,. 33, 35;
183 20, 21, 23, 27

3. LOGARITHMS/SCIENTIFIC NOTATION

186 25; 685 7, 29; 185 8, 12, 35; 684 4, 23; 184 7, 12, 31;
683 8, 25; 183 7, 24

4. TRIGONOMETRIC FUNCTIONS

186 11, 20, 21, 22, 28; 685 16, 20, 23, 28, 32, 35;
185 4, 14, 17, 33 684 9, 15, 17, 27, 29;
683 7, 9, 10, 11, 14, 17, 28; 183 5, 9, 16, 17, 19, 29, 32

5. ABSOLUTE VALUE EQUATIONS/ INEQUALITIES

186 24; 685 9, 31; 185 5; 684 13; 184 24; 683 30

6. AREA OF A TRIANGLE

186 8; 685 4; 185 9; 684 10; 683 13; 682 13

7. CIRCLES/ANGLE MEASURE

186 2, 23; 685 2, 14, 19; 185 10,31; 684 12, 22; 184 8, 9, 10;
683 6, 19; 183 35

8. FUNCTIONS

186 15, 16, 35; 685 3; 185 11, 29, 30; 684 2, 35; 184 3;
683 23

9. SOLUTIONS OF TRIANGLES (INCLUDING AMBIGUOUS CASE*)

685 30, 33; 185 13*, 23; 684 8, 31; 184 14; 683 18;
183 12, 25 26*; 682 18, 29; 182 6*, 34

HIGH SCHOOL MATHEMATICS

COURSE III

REGENTS REVIEW MQ6 - PART I QUESTIONS

10. TRIGONOMETRIC EQUATIONS

1865; 68513; 185 15, 20, 26; 68 11; 182 33

11. RADICALS/EQUATIONS

186 27; 185 16, 27; 684 33, 34; 184 26, 32; 683 3, 15, 22; 183 14; 682 19

12. TRIGONOMETRIC GRAPHS

186 4, 27; 685 24; 185 19; 684 25; 184 20, 25, 30, 34; 683 34; 183 3, 22

13. COMPLEX NUMBERS

186 31, 32; 685 21; 185 21; 684 2, 3; 184 16, 21; 683 1, 12; 183 8, 31

14. STATISTICS

685 25, 27; 185 24, 25; 684 26; 184 23; 683 21; 183 34

15. QUADRATIC EQUATIONS

186 29 33; 685 12, 26; 185 28; 684 24; 184 11, 18; 683 24, 27, 30

16. BINOMIAL THEOREM/ EXPANSION

186 30; 685 34; 684 6; 184 5; 683 31; 183 33

17. SIGMA NOTATION

186 6; 685 5; 684 6; 184 5; 683 2; 183 11

18. PROBABILITY

186 12; 685 11, 22; 684 34; 184 27; 683 26; 183 18

19. EXPONENTIAL EQUATIONS

685 15; 684 16,30; 184 15; 683 16

20. ALGEBRAIC EXPRESSIONS/OPERATIONS

186 7, 9; 685 8; 684 1, 14, 21; 184 15; 683 16

HIGH SCHOOL MATHEMATICS

COURSE III

REGENTS REVIEW MQ6 - PART I QUESTIONS

21. INTERPOLATION

186 13; 185 3; 684 36b

22. INVERSES

186 17; 186 26

23. NORMAL DISTRIBUTION

186 18; 685 27; 684 26; 184 23

REGENTS REVIEW MQ6

PART II QUESTIONS

1. TRIGONOMETRIC GRAPHS

18636; 68538; 18536; 68437; 18436; 68336; 18336; 68236

2. LOGARITHMS/ALGEBRAIC OPERATIONS

18641; 18637b; 68537; 18537; 68436; 18439; 68339; 18338; 68240

3. PROBABILITY/ STATISTICS

18642; 18640; 68540; 68536; 18538; 68441; 18441; 68337; 18339; 68242

4. CIRCLES & ANGLE MEASURES

68542; 18539; 68438; 18437; 68342; 18342

5. SOLUTION OF TRIANGLES (LAWS OF SINES & COSINES)

18638; 68539; 18540; 68442; 18438; 68341; 18340

6. TRANSFORMATIONS/COORDINATE GEOMETRY

18639; 18541; 68439; 18440; 68341; 18341

7. TRIGONOMETRIC IDENTITIES/EQUATIONS

18636a; 68541; 18542; 68440; 18442; 68338; 18337

The University of the State of New York

REGENTS HIGH SCHOOL EXAMINATION

THREE-YEAR SEQUENCE FOR HIGH SCHOOL MATHEMATICS

COURSE III

Wednesday, August 14, 1985 — 8:30 to 11:30 a.m., only

The last page of the booklet is the answer sheet. Fold the last page along the perforations and, slowly and carefully, tear off the answer sheet. Then fill in the heading of your answer sheet.

The "Reference Tables for Mathematics" and a formula sheet which you may need to answer some questions in this examination are stapled in the center of this booklet. Open the booklet and carefully remove the reference tables.

When you have completed the examination, you must sign the statement printed at the end of the answer paper, indicating that you had no unlawful knowledge of the questions or answers prior to the examination and that you have neither given nor received assistance in answering any of the questions during the examination. Your answer paper cannot be accepted if you fail to sign this declaration.

DO NOT OPEN THIS EXAMINATION BOOKLET UNTIL THE SIGNAL IS GIVEN

Part I

Answer 30 questions from this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Write your answers in the spaces provided on the separate answer sheet. Where applicable, answers may be left in terms of π or in radical form. [60]

1 If $f(x) = x^2 - 2x + 3$, find $f(-2)$.

2 Solve for x: $\sqrt{x + 2} - 3 = 0$

3 Express 150° in radian measure.

4 Find the value of $\sum_{k=1}^{4} \frac{12}{k}$

5 In a circle, a central angle of 3 radians intercepts an arc of length 21. Find the length of the radius of the circle.

6 If $8^{2x} = 2^{x+3}$, what is the value of x?

7 A translation maps the origin to the point (5,-3). What is the image of the point (-3,2) under the same translation?

8 For what value of k will the graph of $y = \log_{10} x$ contain the point $(1,k)$?

9 The table below gives a set of measures and their respective frequencies. Find the *mean* of these measures.

Measure (x_i)	Frequency (f_i)
2	2
5	3
7	4
8	1

10 Find the area of triangle ABC if $a = 12$, $b = 15$, and $m\angle C = 30$.

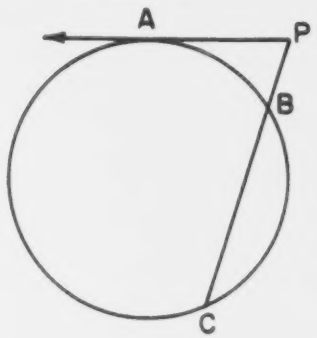
11 Find the value of $(-8)^{\frac{2}{3}}$.

12 Find the value of θ in the interval $0° \leq \theta \leq 360°$ that satisfies the equation $|\sin \theta - 3| = 2$.

13 Find the measure of the smallest positive angle which satisfies the equation $2 \cos^2 \theta + 5 \cos \theta - 3 = 0$.

14 If the equation $9x^2 - 12x + k = 0$ has equal roots, find the value of k.

15 In the diagram below, $\overrightarrow{PA}$ is tangent to the circle at A and $\overline{PBC}$ is a secant. If $PC = 16$ and $CB = 12$, find PA.



16 Find tan 27° 13′ to *four decimal places*.

Directions (17 – 35): For *each* question chosen, write on the separate answer sheet the *numeral* preceding the word or expression that best completes the statement or answers the question.

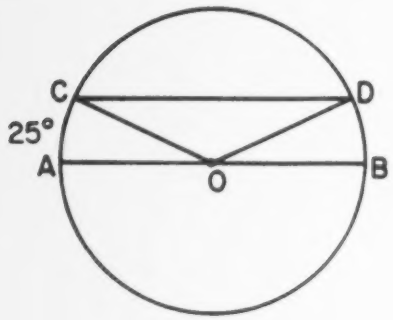
17 What is the total number of lines of symmetry for the letter **H**?

(1) 1 (3) 3
(2) 2 (4) 0

18 The transformation $(x,y) \rightarrow (3x,3y)$ represents

(1) a dilation (3) a reflection
(2) an isometry (4) a translation

19 In the accompanying diagram, chord $\overline{CD}$ is parallel to diameter $\overline{AB}$. If $m\overset{\frown}{AC} = 25$, what is $m\angle COD$?



(1) 25
(2) 65
(3) 130
(4) 155

20 What is the amplitude of the graph of $y = \frac{3}{2} \cos 2x$?

(1) $\frac{3}{2}$
(2) 2
(3) π
(4) 4π

21 Which property is *not* preserved under a line reflection?

(1) collinearity
(2) distance
(3) perpendicularity
(4) orientation

22 What is the last term in the expansion of $(x - 2)^5$?

(1) $-32x$
(2) $32x$
(3) -32
(4) 32

23 If the probability that an event will *not* occur is $\frac{1}{x^2}$, then the probability that the event will occur is represented by

(1) -1
(2) 0
(3) x^2
(4) $\frac{x^2 - 1}{x^2}$

24 Which is an equation of the reflection of the graph of $y = \sin x$ in the x-axis?

(1) $y = \cos x$
(2) $y = \csc x$
(3) $y = -\sin x$
(4) $y = -\sec x$

25 If $\sin x < 0$ and $\tan x > 0$, then x must be in Quadrant

(1) I
(2) II
(3) III
(4) IV

26 The roots of $3x^2 - 4x + 2 = 0$ are

(1) $\frac{1 \pm \sqrt{2}}{3}$
(2) $\frac{2 \pm i\sqrt{2}}{3}$
(3) $\frac{2 \pm \sqrt{10}}{3}$
(4) $4 \pm \frac{i\sqrt{2}}{3}$

27 Which is the value of $\cos(-240°)$?

(1) $-\frac{1}{2}$
(2) $\frac{3}{2}$
(3) $\frac{1}{2}$
(4) $-\frac{3}{2}$

28 The expression $3 \log x - \frac{1}{2} \log y$ is equal to

(1) $\log \frac{x^3}{y^2}$
(2) $\log \frac{x^3}{\sqrt{y}}$
(3) $\log \sqrt{\frac{3x}{y}}$
(4) $\frac{\log 3x}{\frac{1}{2} \log y}$

29 For all values of θ for which the expressions are defined, $(\tan \theta)(\csc \theta)$ is equivalent to

(1) $\cos \theta$
(2) $\cot \theta$
(3) $\tan \theta$
(4) $\sec \theta$

30 Which is an equation of the line formed when the line $y = 3x + 1$ is reflected in the line $y = x$?

(1) $y = 3x - 1$
(2) $y = \frac{x - 1}{3}$
(3) $y = \frac{x}{3} - 1$
(4) $x = y$

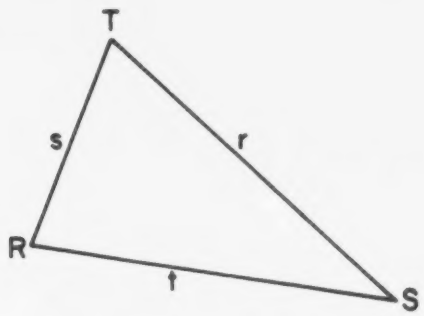
31 For all values of n for which the expressions are defined, $\left(1 + \frac{1}{n}\right) \div \left(\frac{n + 1}{n^2}\right)$ is equivalent to

(1) $\frac{(n + 1)^2}{n^3}$
(2) $\frac{2n}{n + 1}$
(3) $\frac{n^2}{n + 1}$
(4) n

32 A set of scores with a normal distribution has a mean of 50 and a standard deviation of 7. Approximately what percent of the scores fall in the range 36 to 64?

(1) 34%
(2) 68%
(3) 95%
(4) 99%

33 In triangle RST, what is the value of r in terms of R, T, and t?



(1) $r = \frac{tR}{T}$
(2) $r = \frac{t \cdot \sin T}{\sin R}$
(3) $r = \frac{\sin T}{t \cdot \sin R}$
(4) $r = \frac{t \cdot \sin R}{\sin T}$

34 What are the coordinates of M', the image of $M(2,4)$, after a counterclockwise rotation of 90° about the origin?

(1) (−2,4)
(2) (−2,−4)
(3) (−4,2)
(4) (−4,−2)

35 In triangle ABC, $a = 2$, $b = 3$, and $c = 4$. What is the value of $\cos C$?

(1) $-\frac{1}{4}$
(2) $\frac{1}{4}$
(3) $\frac{1}{16}$
(4) $-\frac{1}{16}$

Answers to the following questions are to be written on paper provided by the school.

Part II

Answer four questions from this part. Show all work unless otherwise directed. [40]

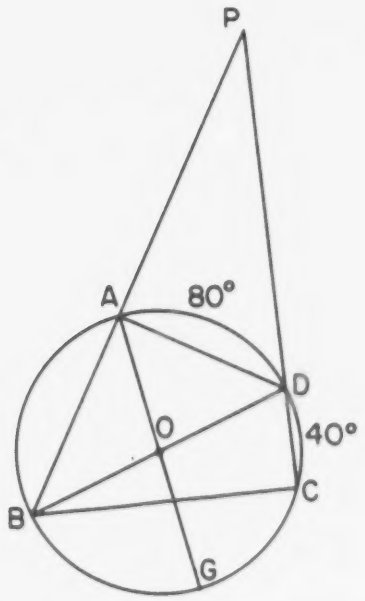
36 *a* On the same set of axes, sketch the graphs of $y = 2 \sin x$ and $y = x^2$, for values of x in the interval $-\pi \leq x \leq \pi$. [8]

b Using the graph made in part *a*, how many values in the interval $-\pi \leq x \leq \pi$ satisfy the equation $x^2 = 2 \sin x$? [2]

37 *a* Using logarithms, solve the equation $5^x = 17$ for x to the *nearest tenth*. [5]

b Express the roots of the equation $x^2 = 2x - 5$ in $a + bi$ form. [5]

38 Quadrilateral $ABCD$ is inscribed in circle O, $\overline{BD}$ and $\overline{AG}$ are diameters, $\overline{PAB}$ and $\overline{PDC}$ are secants, $m\widehat{AD} = 80$, and $m\widehat{DC} = 40$.



Find:

a $m\widehat{AB}$ [2]
b $m\angle BCD$ [2]
c $m\angle BOG$ [2]
d $m\angle P$ [2]
e $m\angle BAG$ [2]

39 *a* Sketch and label the graph of $y = 2^x$. [4]

b The graph of $y = 2^x$ is subject to each of these transformations:

(1) reflection in the y-axis
(2) reflection in the line $y = x$
(3) translation: $(x,y) \rightarrow (x,y+1)$

On your answer paper, write the numerals (1), (2), and (3). Next to the appropriate numeral, write the letter of the equation, *chosen from the list below*, that best describes the image of $y = 2^x$ under each of the given transformations. [2,2,2]

Equations

(a) $y = \log_2 x$
(b) $y = -2^x$
(c) $y = 2^{-x}$
(d) $y = 2^x + 1$

40 The table below shows the height in inches of ten girls on a basketball team.

Height (x_i)	Frequency (f_i)
62	2
66	1
68	2
72	3
74	2

a Find the mode. [1]
b Find the median. [2]
c Find the standard deviation of these heights to the *nearest tenth*. [7]

☞ GO RIGHT ON TO THE NEXT PAGE.

41 Two adjacent sides of a parallelogram are 10 and 12 centimeters. The angle included between these sides has a measure of 60°

a Find, to the *nearest centimeter*, the length of the shorter diagonal of the parallelogram. [6]

b What is the area of the parallelogram to the *nearest square centimeter*? [4]

42 *a* Find, to the *nearest degree*, all values of θ in the interval $0° \leq \theta < 360°$ which satisfy the equation $6 \sin^2 \theta = 1 - \sin \theta$. [6]

b For all values of θ for which the expressions are defined, prove that the following is an identity:

$$\tan \theta + \cot \theta = \sec \theta \csc \theta$$ [4]

The University of the State of New York

REGENTS HIGH SCHOOL EXAMINATION

SEQUENTIAL MATH — COURSE III

Wednesday, August 14, 1985 – 8:30 to 11:30 a.m., only

Part I Score	
Part II Score	
Total Score	
Rater's Initials	

ANSWER SHEET

Pupil ..... Teacher .....

School ..... Grade .....

Your answers to Part I should be recorded on this answer sheet.

Part I

Answer 30 questions from this part.

1	11	21	31 .....
2	12	22	32 .....
3	13	23	33 .....
4	14	24	34 .....
5	15	25	35 .....
6	16	26	
7	17	27	
8	18	28	
9	19	29	
10	20	30	

Your answers for Part II should be placed on paper provided by the school.

The declaration below should be signed when you have completed the examination.

I do hereby affirm, at the close of this examination, that I had no unlawful knowledge of the questions or answers prior to the examination, and that I have neither given nor received assistance in answering any of the questions during the examination.

Signature

FOR TEACHERS ONLY

SCORING KEY

THREE-YEAR SEQUENCE FOR HIGH SCHOOL MATHEMATICS

COURSE III

Wednesday, August 14, 1985 – 8:30 to 11:30 a.m., only

Use only *red* ink or *red* pencil in rating Regents papers. Do not attempt to *correct* the pupil's work by making insertions or changes of any kind. Use checkmarks to indicate pupil errors.

Unless otherwise specified, mathematically correct variations in the answers will be allowed. Units need not be given when the wording of the questions allows such omissions.

Part I

Allow a total of 60 credits, 2 credits for each of 30 of the following. [If more than 30 are answered, only the first 30 answered should be considered.] Allow no partial credit. For questions 17 – 35, allow credit if the pupil has written the correct answer instead of the numeral 1, 2, 3, or 4.

(1) 11

(2) 7

(3) $\frac{5\pi}{6}$

(4) 25

(5) 7

(6) 1

(7) (2,–1)

(8) 0

(9) 5.5

(10) 45

(11) 4

(12) 90°

(13) 60° *or* $\frac{\pi}{3}$

(14) 4

(15) 8

(16) 0.5143

(17) 2

(18) 1

(19) 3

(20) 1

(21) 4

(22) 3

(23) 4

(24) 3

(25) 3

(26) 2

(27) 1

(28) 2

(29) 4

(30) 2

(31) 4

(32) 3

(33) 4

(34) 3

(35) 1

[OVER]

Part II

Please refer to the Department's pamphlet *Suggestions on the Rating of Regents Examination Papers in Mathematics*. Care should be exercised in making deductions as to whether the error is purely a mechanical one or due to a violation of some principle. A mechanical error generally should receive a deduction of 10 percent, while an error due to a violation of some cardinal principle should receive a deduction ranging from 30 percent to 50 percent, depending on the relative importance of the principle in the solution of the problem.

(36) *b* 2 [2]

(37) *a* 1.8 [5]
b $1 \pm 2i$ [5]

(38) *a* 100 [2]
b 90 [2]
c 80 [2]
d 30 [2]
e 40 [2]

(39) *b* (1) c [2]
(2) a [2]
(3) d [2]

(40) *a* 72 [1]
b 70 [2]
c 4.3 [7]

(41) *a* 11 [6]
b 104 [4]

(42) *a* 19°, 161°, 210°, 330° [6]

The University of the State of New York

REGENTS HIGH SCHOOL EXAMINATION

THREE-YEAR SEQUENCE FOR HIGH SCHOOL MATHEMATICS

COURSE III

Monday, June 23, 1986 — 1:15 to 4:15 p.m., only

The last page of the booklet is the answer sheet. Fold the last page along the perforations and, slowly and carefully, tear off the answer sheet. Then fill in the heading of your answer sheet.

The "Reference Tables for Mathematics" and a formula sheet which you may need to answer some questions in this examination are stapled in the center of this booklet. Open the booklet and carefully remove the reference tables.

When you have completed the examination, you must sign the statement printed at the end of the answer paper, indicating that you had no unlawful knowledge of the questions or answers prior to the examination and that you have neither given nor received assistance in answering any of the questions during the examination. Your answer paper cannot be accepted if you fail to sign this declaration.

DO NOT OPEN THIS EXAMINATION BOOKLET UNTIL THE SIGNAL IS GIVEN

Part I

Answer 30 questions from this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Write your answers in the spaces provided on the separate answer sheet. Where applicable, answers may be left in terms of π or in radical form. [60]

1 Express 105° in radian measure.

2 If $f(x) = 3 - x^2$, find $f(-2)$.

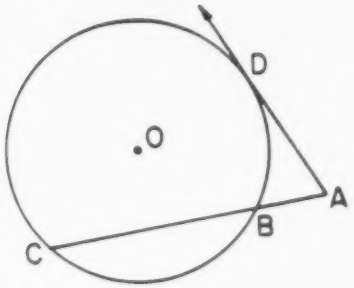
3 Evaluate: $\sum_{x=1}^{3} (2x - 1)$

4 Solve for x: $\sqrt{3x - 5} = 5$

5 If point P with coordinates (a,b) is reflected in the line $y = x$, what are the coordinates of the image of P?

6 In $\triangle ABC$, $a = 12$, $b = 8$, and $m\angle C = 30$. Find the area of $\triangle ABC$.

7 In the accompanying figure, $\overrightarrow{AD}$ is tangent to circle O at D and $\overline{ABC}$ is a secant. If $AD = 4$ and $AC = 8$, find AB.



8 Express as a fraction in *simplest form*:

$$\frac{\frac{1}{ab}}{\frac{1}{a} - \frac{1}{b}}$$

9 In $\triangle ABC$, $b = 12$, $c = 8$, and $\sin B = \frac{1}{2}$. Find the value of $\sin C$.

10 Solve for x: $(5 - 2i) - (x + 4i) = 7 - 6i$

11 If $g(x) = x^{-\frac{1}{2}}$, find $g(4)$.

12 Two tangents to a circle from an external point intercept a major arc of 300°. Find the number of degrees in the angle formed by the two tangents.

13 Express $\frac{3}{3 - \sqrt{3}}$ as a fraction with a rational denominator.

14 If $10^{3.3331} = 3590$, find the value of $10^{0.3331}$

15 In $\triangle ABC$, $a = 3$, $b = 5$, and $m\angle C = 120$. Find the value of c.

16 Find the value of cos 32° 32′ to *four decimal places*.

17 Express sin (−215°) as a function of a positive acute angle.

18 In a circle, an arc of length 10 is intercepted by a central angle of $\frac{2}{3}$ radian. Find the radius of the circle.

19 What are the coordinates of P', the image of the point $P(2,3)$, under the transformation $r_{x=4} \circ r_{x\text{-axis}}$?

Directions (20 – 35): For *each* question chosen, write on the separate answer sheet the *numeral* preceding the word or expression that best completes the statement or answers the question.

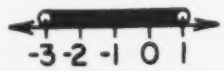
20 Which symbol has 90° rotational symmetry?

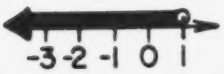
(1) X (3) S
(2) H (4) 8

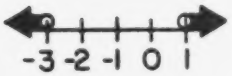
21 If translation T maps point $A(-3,1)$ onto point $A'(5,5)$, which is the translation T?

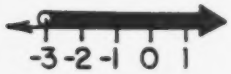
(1) $T_{2,4}$ (3) $T_{8,6}$
(2) $T_{2,6}$ (4) $T_{8,4}$

22 Which graph represents the solution set of $|x + 1| < 2$?

(1) 

(2) 

(3) 

(4) 

23 For all values of x for which the expressions are defined, $\sec x - \tan x$ is equivalent to

(1) 1 (3) $\frac{1 - \sin x}{\cos x}$

(2) $\cos x - \cot x$ (4) $\frac{\cos x - \sin^2 x}{\sin x \cos x}$

24 On a standardized test, Cathy had a score of 74, which was exactly 1 standard deviation below the mean. If the standard deviation for the test is 6, what is the mean score for the test?

(1) 68 (3) 77
(2) 71 (4) 80

25 Which letter has point symmetry?

(1) **A** (3) **N**
(2) **B** (4) **T**

26 The roots of the equation $ax^2 + 4x + 2 = 0$ are real and equal when a is equal to

(1) 1 (3) 3
(2) 2 (4) 4

27 In the interval $0° \leq x < 360°$, $\sin x = \cos x$ when x is

(1) 45°, only (3) 135° and 315°
(2) 45° and 225° (4) 225°, only

28 If the terminal side of angle θ passes through the point $(-4,3)$, what is the value of $\cos \theta$?

(1) $\frac{3}{5}$ (3) $\frac{4}{5}$

(2) $-\frac{3}{5}$ (4) $-\frac{4}{5}$

29 In a family of six children, what is the probability that there will be exactly one male child?

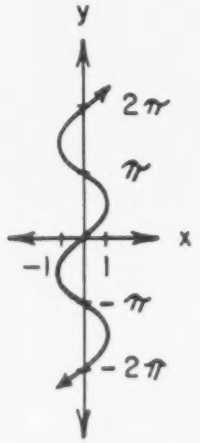
(1) $\frac{6}{64}$ (3) $\frac{32}{64}$

(2) $\frac{7}{64}$ (4) $\frac{58}{64}$

30 The expression $\sin(90° - \theta)$ is equivalent to

(1) $\cos \theta$ (3) $-\cos \theta$
(2) $\sin \theta$ (4) $-\sin \theta$

31 Which is an equation of the graph shown below?



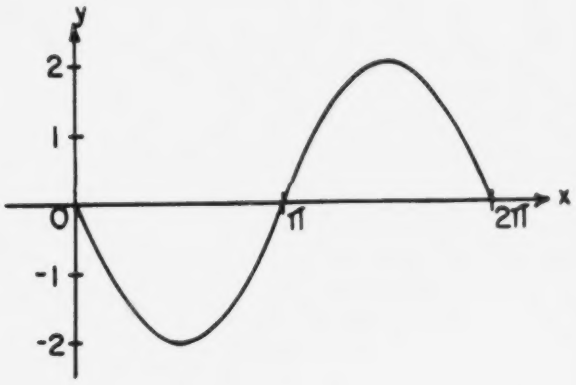
(1) $y = \arcsin x$ (3) $y = \sec x$
(2) $y = \arccos x$ (4) $y = \csc x$

32 If $\tan x = -\frac{2}{3}$ and angle x lies in the second quadrant, what is the value of $\cos x$?

(1) $\frac{3\sqrt{5}}{5}$ (3) $\frac{3\sqrt{13}}{13}$

(2) $-\frac{3\sqrt{5}}{5}$ (4) $-\frac{3\sqrt{13}}{13}$

33 Which is an equation of the graph shown below?



(1) $y = \sin 2x$
(2) $y = -\sin 2x$
(3) $y = -2 \sin x$
(4) $y = 2 \sin x$

34 Given the relation $R = \{(-2,3),(a,4),(1,9),(0,7)\}$. Which replacement for a makes this relation a function?

(1) 1
(2) -2
(3) 0
(4) 4

35 What is the inverse of the function $y = 2x + 3$?

(1) $x = \frac{1}{2}y - \frac{3}{2}$
(2) $y = \frac{1}{2}x - \frac{3}{2}$
(3) $y = 2x - 3$
(4) $x = -2y - 3$

Formulas

Pythagorean and Quotient Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A + 1 = \sec^2 A$$
$$\cot^2 A + 1 = \csc^2 A$$

$$\tan A = \frac{\sin A}{\cos A}$$
$$\cot A = \frac{\cos A}{\sin A}$$

Functions of the Sum of Two Angles

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Functions of the Difference of Two Angles

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Functions of the Double Angle

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 2 \cos^2 A - 1$$
$$\cos 2A = 1 - 2 \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Functions of the Half Angle

$$\sin \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \tfrac{1}{2}A = \pm \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan \tfrac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Area of Triangle

$$K = \tfrac{1}{2}ab \sin C$$

Standard Deviation

$$\text{S.D.} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\bar{x} - x_i)^2}$$

Table A: Common Logarithms of Numbers*

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

* This table gives the mantissas of numbers with the decimal point omitted in each case. Characteristics are determined from the numbers by inspection.

180

Table A: Common Logarithms of Numbers*

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

* This table gives the mantissas of numbers with the decimal point omitted in each case. Characteristics are determined from the numbers by inspection.

Table B: Values of Trigonometric Functions

Angle	Sin	Cos	Tan	Cot	
0° 00′	.0000	1.0000	.0000	—	90° 00′
10	.0029	1.0000	.0029	343.77	50
20	.0058	1.0000	.0058	171.89	40
30	.0087	1.0000	.0087	114.59	30
40	.0116	.9999	.0116	85.940	20
50	.0145	.9999	.0145	68.750	10
1° 00′	.0175	.9998	.0175	57.290	89° 00′
10	.0204	.9998	.0204	49.104	50
20	.0233	.9997	.0233	42.964	40
30	.0262	.9997	.0262	38.188	30
40	.0291	.9996	.0291	34.368	20
50	.0320	.9995	.0320	31.242	10
2° 00′	.0349	.9994	.0349	28.636	88° 00′
10	.0378	.9993	.0378	26.432	50
20	.0407	.9992	.0407	24.542	40
30	.0436	.9990	.0437	22.904	30
40	.0465	.9989	.0466	21.470	20
50	.0494	.9988	.0495	20.206	10
3° 00′	.0523	.9986	.0524	19.081	87° 00′
10	.0552	.9985	.0553	18.075	50
20	.0581	.9983	.0582	17.169	40
30	.0610	.9981	.0612	16.350	30
40	.0640	.9980	.0641	15.605	20
50	.0669	.9978	.0670	14.924	10
4° 00′	.0698	.9976	.0699	14.301	86° 00′
10	.0727	.9974	.0729	13.727	50
20	.0756	.9971	.0758	13.197	40
30	.0785	.9969	.0787	12.706	30
40	.0814	.9967	.0816	12.251	20
50	.0843	.9964	.0846	11.826	10
5° 00′	.0872	.9962	.0875	11.430	85° 00′
10	.0901	.9959	.0904	11.059	50
20	.0929	.9957	.0934	10.712	40
30	.0958	.9954	.0963	10.385	30
40	.0987	.9951	.0992	10.078	20
50	.1016	.9948	.1022	9.7882	10
6° 00′	.1045	.9945	.1051	9.5144	84° 00′
10	.1074	.9942	.1080	9.2553	50
20	.1103	.9939	.1110	9.0098	40
30	.1132	.9936	.1139	8.7769	30
40	.1161	.9932	.1169	8.5555	20
50	.1190	.9929	.1198	8.3450	10
7° 00′	.1219	.9925	.1228	8.1443	83° 00′
10	.1248	.9922	.1257	7.9530	50
20	.1276	.9918	.1287	7.7704	40
30	.1305	.9914	.1317	7.5958	30
40	.1334	.9911	.1346	7.4287	20
50	.1363	.9907	.1376	7.2687	10
8° 00′	.1392	.9903	.1405	7.1154	82° 00′
10	.1421	.9899	.1435	6.9682	50
20	.1449	.9894	.1465	6.8269	40
30	.1478	.9890	.1495	6.6912	30
40	.1507	.9886	.1524	6.5606	20
50	.1536	.9881	.1554	6.4348	10
9° 00′	.1564	.9877	.1584	6.3138	81° 00′
10	.1593	.9872	.1614	6.1970	50
20	.1622	.9868	.1644	6.0844	40
30	.1650	.9863	.1673	5.9758	30
40	.1679	.9858	.1703	5.8708	20
50	.1708	.9853	.1733	5.7694	10
10° 00′	.1736	.9848	.1763	5.6713	80° 00′
10	.1765	.9843	.1793	5.5764	50
20	.1794	.9838	.1823	5.4845	40
30	.1822	.9833	.1853	5.3955	30
40	.1851	.9827	.1883	5.3093	20
50	.1880	.9822	.1914	5.2257	10
11° 00′	.1908	.9816	.1944	5.1446	79° 00′
10	.1937	.9811	.1974	5.0658	50
20	.1965	.9805	.2004	4.9894	40
30	.1994	.9799	.2035	4.9152	30
40	.2022	.9793	.2065	4.8430	20
50	.2051	.9787	.2095	4.7729	10
12° 00′	.2079	.9781	.2126	4.7046	78° 00′
	Cos	Sin	Cot	Tan	Angle

Angle	Sin	Cos	Tan	Cot	
12° 00′	.2079	.9781	.2126	4.7046	78° 00′
10	.2108	.9775	.2156	4.6382	50
20	.2136	.9769	.2186	4.5736	40
30	.2164	.9763	.2217	4.5107	30
40	.2193	.9757	.2247	4.4494	20
50	.2221	.9750	.2278	4.3897	10
13° 00′	.2250	.9744	.2309	4.3315	77° 00′
10	.2278	.9737	.2339	4.2747	50
20	.2306	.9730	.2370	4.2193	40
30	.2334	.9724	.2401	4.1653	30
40	.2363	.9717	.2432	4.1126	20
50	.2391	.9710	.2462	4.0611	10
14° 00′	.2419	.9703	.2493	4.0108	76° 00′
10	.2447	.9696	.2524	3.9617	50
20	.2476	.9689	.2555	3.9136	40
30	.2504	.9681	.2586	3.8667	30
40	.2532	.9674	.2617	3.8208	20
50	.2560	.9667	.2648	3.7760	10
15° 00′	.2588	.9659	.2679	3.7321	75° 00′
10	.2616	.9652	.2711	3.6891	50
20	.2644	.9644	.2742	3.6470	40
30	.2672	.9636	.2773	3.6059	30
40	.2700	.9628	.2805	3.5656	20
50	.2728	.9621	.2836	3.5261	10
16° 00′	.2756	.9613	.2867	3.4874	74° 00′
10	.2784	.9605	.2899	3.4495	50
20	.2812	.9596	.2931	3.4124	40
30	.2840	.9588	.2962	3.3759	30
40	.2868	.9580	.2994	3.3402	20
50	.2896	.9572	.3026	3.3052	10
17° 00′	.2924	.9563	.3057	3.2709	73° 00′
10	.2952	.9555	.3089	3.2371	50
20	.2979	.9546	.3121	3.2041	40
30	.3007	.9537	.3153	3.1716	30
40	.3035	.9528	.3185	3.1397	20
50	.3062	.9520	.3217	3.1084	10
18° 00′	.3090	.9511	.3249	3.0777	72° 00′
10	.3118	.9502	.3281	3.0475	50
20	.3145	.9492	.3314	3.0178	40
30	.3173	.9483	.3346	2.9887	30
40	.3201	.9474	.3378	2.9600	20
50	.3228	.9465	.3411	2.9319	10
19° 00′	.3256	.9455	.3443	2.9042	71° 00′
10	.3283	.9446	.3476	2.8770	50
20	.3311	.9436	.3508	2.8502	40
30	.3338	.9426	.3541	2.8239	30
40	.3365	.9417	.3574	2.7980	20
50	.3393	.9407	.3607	2.7725	10
20° 00′	.3420	.9397	.3640	2.7475	70° 00′
10	.3448	.9387	.3673	2.7228	50
20	.3475	.9377	.3706	2.6985	40
30	.3502	.9367	.3739	2.6746	30
40	.3529	.9356	.3772	2.6511	20
50	.3557	.9346	.3805	2.6279	10
21° 00′	.3584	.9336	.3839	2.6051	69° 00′
10	.3611	.9325	.3872	2.5826	50
20	.3638	.9315	.3906	2.5605	40
30	.3665	.9304	.3939	2.5386	30
40	.3692	.9293	.3973	2.5172	20
50	.3719	.9283	.4006	2.4960	10
22° 00′	.3746	.9272	.4040	2.4751	68° 00′
10	.3773	.9261	.4074	2.4545	50
20	.3800	.9250	.4108	2.4342	40
30	.3827	.9239	.4142	2.4142	30
40	.3854	.9228	.4176	2.3945	20
50	.3881	.9216	.4210	2.3750	10
23° 00′	.3907	.9205	.4245	2.3559	67° 00′
10	.3934	.9194	.4279	2.3369	50
20	.3961	.9182	.4314	2.3183	40
30	.3987	.9171	.4348	2.2998	30
40	.4014	.9159	.4383	2.2817	20
50	.4041	.9147	.4417	2.2637	10
24° 00′	.4067	.9135	.4452	2.2460	66° 00′
	Cos	Sin	Cot	Tan	Angle

Table B: Values of Trigonometric Functions

Angle	Sin	Cos	Tan	Cot	
24° 00′	.4067	.9135	.4452	2.2460	66° 00′
10	.4094	.9124	.4487	2.2286	50
20	.4120	.9112	.4522	2.2113	40
30	.4147	.9100	.4557	2.1943	30
40	.4173	.9088	.4592	2.1775	20
50	.4200	.9075	.4628	2.1609	10
25° 00′	.4226	.9063	.4663	2.1445	65° 00′
10	.4253	.9051	.4699	2.1283	50
20	.4279	.9038	.4734	2.1123	40
30	.4305	.9026	.4770	2.0965	30
40	.4331	.9013	.4806	2.0809	20
50	.4358	.9001	.4841	2.0655	10
26° 00′	.4384	.8988	.4877	2.0503	64° 00′
10	.4410	.8975	.4913	2.0353	50
20	.4436	.8962	.4950	2.0204	40
30	.4462	.8949	.4986	2.0057	30
40	.4488	.8936	.5022	1.9912	20
50	.4514	.8923	.5059	1.9768	10
27° 00′	.4540	.8910	.5095	1.9626	63° 00′
10	.4566	.8897	.5132	1.9486	50
20	.4592	.8884	.5169	1.9347	40
30	.4617	.8870	.5206	1.9210	30
40	.4643	.8857	.5243	1.9074	20
50	.4669	.8843	.5280	1.8940	10
28° 00′	.4695	.8829	.5317	1.8807	62° 00′
10	.4720	.8816	.5354	1.8676	50
20	.4746	.8802	.5392	1.8546	40
30	.4772	.8788	.5430	1.8418	30
40	.4797	.8774	.5467	1.8291	20
50	.4823	.8760	.5505	1.8165	10
29° 00′	.4848	.8746	.5543	1.8040	61° 00′
10	.4874	.8732	.5581	1.7917	50
20	.4899	.8718	.5619	1.7796	40
30	.4924	.8704	.5658	1.7675	30
40	.4950	.8689	.5696	1.7556	20
50	.4975	.8675	.5735	1.7437	10
30° 00′	.5000	.8660	.5774	1.7321	60° 00′
10	.5025	.8646	.5812	1.7205	50
20	.5050	.8631	.5851	1.7090	40
30	.5075	.8616	.5890	1.6977	30
40	.5100	.8601	.5930	1.6864	20
50	.5125	.8587	.5969	1.6753	10
31° 00′	.5150	.8572	.6009	1.6643	59° 00′
10	.5175	.8557	.6048	1.6534	50
20	.5200	.8542	.6088	1.6426	40
30	.5225	.8526	.6128	1.6319	30
40	.5250	.8511	.6168	1.6212	20
50	.5275	.8496	.6208	1.6107	10
32° 00′	.5299	.8480	.6249	1.6003	58° 00′
10	.5324	.8465	.6289	1.5900	50
20	.5348	.8450	.6330	1.5798	40
30	.5373	.8434	.6371	1.5697	30
40	.5398	.8418	.6412	1.5597	20
50	.5422	.8403	.6453	1.5497	10
33° 00′	.5446	.8387	.6494	1.5399	57° 00′
10	.5471	.8371	.6536	1.5301	50
20	.5495	.8355	.6577	1.5204	40
30	.5519	.8339	.6619	1.5108	30
40	.5544	.8323	.6661	1.5013	20
50	.5568	.8307	.6703	1.4919	10
34° 00′	.5592	.8290	.6745	1.4826	56° 00′
10	.5616	.8274	.6787	1.4733	50
20	.5640	.8258	.6830	1.4641	40
30	.5664	.8241	.6873	1.4550	30
40	.5688	.8225	.6916	1.4460	20
50	.5712	.8208	.6959	1.4370	10
35° 00′	.5736	.8192	.7002	1.4281	55° 00′
10	.5760	.8175	.7046	1.4193	50
20	.5783	.8158	.7089	1.4106	40
30	.5807	.8141	.7133	1.4019	30
40	.5831	.8124	.7177	1.3934	20
50	.5854	.8107	.7221	1.3848	10
36° 00′	.5878	.8090	.7265	1.3764	54° 00′
	Cos	Sin	Cot	Tan	Angle

Angle	Sin	Cos	Tan	Cot	
36° 00′	.5878	.8090	.7265	1.3764	54° 00′
10	.5901	.8073	.7310	1.3680	50
20	.5925	.8056	.7355	1.3597	40
30	.5948	.8039	.7400	1.3514	30
40	.5972	.8021	.7445	1.3432	20
50	.5995	.8004	.7490	1.3351	10
37° 00′	.6018	.7986	.7536	1.3270	53° 00′
10	.6041	.7969	.7581	1.3190	50
20	.6065	.7951	.7627	1.3111	40
30	.6088	.7934	.7673	1.3032	30
40	.6111	.7916	.7720	1.2954	20
50	.6134	.7898	.7766	1.2876	10
38° 00′	.6157	.7880	.7813	1.2799	52° 00′
10	.6180	.7862	.7860	1.2723	50
20	.6202	.7844	.7907	1.2647	40
30	.6225	.7826	.7954	1.2572	30
40	.6248	.7808	.8002	1.2497	20
50	.6271	.7790	.8050	1.2423	10
39° 00′	.6293	.7771	.8098	1.2349	51° 00′
10	.6316	.7753	.8146	1.2276	50
20	.6338	.7735	.8195	1.2203	40
30	.6361	.7716	.8243	1.2131	30
40	.6383	.7698	.8292	1.2059	20
50	.6406	.7679	.8342	1.1988	10
40° 00′	.6428	.7660	.8391	1.1918	50° 00′
10	.6450	.7642	.8441	1.1847	50
20	.6472	.7623	.8491	1.1778	40
30	.6494	.7604	.8541	1.1708	30
40	.6517	.7585	.8591	1.1640	20
50	.6539	.7566	.8642	1.1571	10
41° 00′	.6561	.7547	.8693	1.1504	49° 00′
10	.6583	.7528	.8744	1.1436	50
20	.6604	.7509	.8796	1.1369	40
30	.6626	.7490	.8847	1.1303	30
40	.6648	.7470	.8899	1.1237	20
50	.6670	.7451	.8952	1.1171	10
42° 00′	.6691	.7431	.9004	1.1106	48° 00′
10	.6713	.7412	.9057	1.1041	50
20	.6734	.7392	.9110	1.0977	40
30	.6756	.7373	.9163	1.0913	30
40	.6777	.7353	.9217	1.0850	20
50	.6799	.7333	.9271	1.0786	10
43° 00′	.6820	.7314	.9325	1.0724	47° 00′
10	.6841	.7294	.9380	1.0661	50
20	.6862	.7274	.9435	1.0599	40
30	.6884	.7254	.9490	1.0538	30
40	.6905	.7234	.9545	1.0477	20
50	.6926	.7214	.9601	1.0416	10
44° 00′	.6947	.7193	.9657	1.0355	46° 00′
10	.6967	.7173	.9713	1.0295	50
20	.6988	.7153	.9770	1.0235	40
30	.7009	.7133	.9827	1.0176	30
40	.7030	.7112	.9884	1.0117	20
50	.7050	.7092	.9942	1.0058	10
45° 00′	.7071	.7071	1.0000	1.0000	45° 00′
	Cos	Sin	Cot	Tan	Angle

Table C: Logarithms of Trigonometric Functions*

Angle	L Sin	L Cos	L Tan	L Cot	
0° 00′	—	10.0000	—	—	90° 00′
10	7.4637	10.0000	7.4637	12.5363	50
20	7.7648	10.0000	7.7648	12.2352	40
30	7.9408	10.0000	7.9409	12.0591	30
40	8.0658	10.0000	8.0658	11.9342	20
50	8.1627	10.0000	8.1627	11.8373	10
1° 00′	8.2419	9.9999	8.2419	11.7581	89° 00′
10	8.3088	9.9999	8.3089	11.6911	50
20	8.3668	9.9999	8.3669	11.6331	40
30	8.4179	9.9999	8.4181	11.5819	30
40	8.4637	9.9998	8.4638	11.5362	20
50	8.5050	9.9998	8.5053	11.4947	10
2° 00′	8.5428	9.9997	8.5431	11.4569	88° 00′
10	8.5776	9.9997	8.5779	11.4221	50
20	8.6097	9.9996	8.6101	11.3899	40
30	8.6397	9.9996	8.6401	11.3599	30
40	8.6677	9.9995	8.6682	11.3318	20
50	8.6940	9.9995	8.6945	11.3055	10
3° 00′	8.7188	9.9994	8.7194	11.2806	87° 00′
10	8.7423	9.9993	8.7429	11.2571	50
20	8.7645	9.9993	8.7652	11.2348	40
30	8.7857	9.9992	8.7865	11.2135	30
40	8.8059	9.9991	8.8067	11.1933	20
50	8.8251	9.9990	8.8261	11.1739	10
4° 00′	8.8436	9.9989	8.8446	11.1554	86° 00′
10	8.8613	9.9989	8.8624	11.1376	50
20	8.8783	9.9988	8.8795	11.1205	40
30	8.8946	9.9987	8.8960	11.1040	30
40	8.9104	9.9986	8.9118	11.0882	20
50	8.9256	9.9985	8.9272	11.0728	10
5° 00′	8.9403	9.9983	8.9420	11.0580	85° 00′
10	8.9545	9.9982	8.9563	11.0437	50
20	8.9682	9.9981	8.9701	11.0299	40
30	8.9816	9.9980	8.9836	11.0164	30
40	8.9945	9.9979	8.9966	11.0034	20
50	9.0070	9.9977	9.0093	10.9907	10
6° 00′	9.0192	9.9976	9.0216	10.9784	84° 00′
10	9.0311	9.9975	9.0336	10.9664	50
20	9.0426	9.9973	9.0453	10.9547	40
30	9.0539	9.9972	9.0567	10.9433	30
40	9.0648	9.9971	9.0678	10.9322	20
50	9.0755	9.9969	9.0786	10.9214	10
7° 00′	9.0859	9.9968	9.0891	10.9109	83° 00′
10	9.0961	9.9966	9.0995	10.9005	50
20	9.1060	9.9964	9.1096	10.8904	40
30	9.1157	9.9963	9.1194	10.8806	30
40	9.1252	9.9961	9.1291	10.8709	20
50	9.1345	9.9959	9.1385	10.8615	10
8° 00′	9.1436	9.9958	9.1478	10.8522	82° 00′
10	9.1525	9.9956	9.1569	10.8431	50
20	9.1612	9.9954	9.1658	10.8342	40
30	9.1697	9.9952	9.1745	10.8255	30
40	9.1781	9.9950	9.1831	10.8169	20
50	9.1863	9.9948	9.1915	10.8085	10
9° 00′	9.1943	9.9946	9.1997	10.8003	81° 00′
10	9.2022	9.9944	9.2078	10.7922	50
20	9.2100	9.9942	9.2158	10.7842	40
30	9.2176	9.9940	9.2236	10.7764	30
40	9.2251	9.9938	9.2313	10.7687	20
50	9.2324	9.9936	9.2389	10.7611	10
10° 00′	9.2397	9.9934	9.2463	10.7537	80° 00′
10	9.2468	9.9931	9.2536	10.7464	50
20	9.2538	9.9929	9.2609	10.7391	40
30	9.2606	9.9927	9.2680	10.7320	30
40	9.2674	9.9924	9.2750	10.7250	20
50	9.2740	9.9922	9.2819	10.7181	10
11° 00′	9.2806	9.9919	9.2887	10.7113	79° 00′
10	9.2870	9.9917	9.2953	10.7047	50
20	9.2934	9.9914	9.3020	10.6980	40
30	9.2997	9.9912	9.3085	10.6915	30
40	9.3058	9.9909	9.3149	10.6851	20
50	9.3119	9.9907	9.3212	10.6788	10
12° 00′	9.3179	9.9904	9.3273	10.6725	78° 00′
	L Cos	L Sin	L Cot	L Tan	Angle

Angle	L Sin	L Cos	L Tan	L Cot	
12° 00′	9.3179	9.9904	9.3273	10.6725	78° 00′
10	9.3238	9.9901	9.3336	10.6664	50
20	9.3296	9.9899	9.3397	10.6603	40
30	9.3353	9.9896	9.3458	10.6542	30
40	9.3410	9.9893	9.3517	10.6483	20
50	9.3466	9.9890	9.3576	10.6424	10
13° 00′	9.3521	9.9887	9.3634	10.6366	77° 00′
10	9.3575	9.9884	9.3691	10.6309	50
20	9.3629	9.9881	9.3748	10.6252	40
30	9.3682	9.9878	9.3804	10.6196	30
40	9.3734	9.9875	9.3859	10.6141	20
50	9.3786	9.9872	9.3914	10.6086	10
14° 00′	9.3837	9.9869	9.3968	10.6032	76° 00′
10	9.3887	9.9866	9.4021	10.5979	50
20	9.3937	9.9863	9.4074	10.5926	40
30	9.3986	9.9859	9.4127	10.5873	30
40	9.4035	9.9856	9.4178	10.5822	20
50	9.4083	9.9853	9.4230	10.5770	10
15° 00′	9.4130	9.9849	9.4281	10.5719	75° 00′
10	9.4177	9.9846	9.4331	10.5669	50
20	9.4223	9.9843	9.4381	10.5619	40
30	9.4269	9.9839	9.4430	10.5570	30
40	9.4314	9.9836	9.4479	10.5521	20
50	9.4359	9.9832	9.4527	10.5473	10
16° 00′	9.4403	9.9828	9.4575	10.5425	74° 00′
10	9.4447	9.9825	9.4622	10.5378	50
20	9.4491	9.9821	9.4669	10.5331	40
30	9.4533	9.9817	9.4716	10.5284	30
40	9.4576	9.9814	9.4762	10.5238	20
50	9.4618	9.9810	9.4808	10.5192	10
17° 00′	9.4659	9.9806	9.4853	10.5147	73° 00′
10	9.4700	9.9802	9.4898	10.5102	50
20	9.4741	9.9798	9.4943	10.5057	40
30	9.4781	9.9794	9.4987	10.5013	30
40	9.4821	9.9790	9.5031	10.4969	20
50	9.4861	9.9786	9.5075	10.4925	10
18° 00′	9.4900	9.9782	9.5118	10.4882	72° 00′
10	9.4939	9.9778	9.5161	10.4839	50
20	9.4977	9.9774	9.5203	10.4797	40
30	9.5015	9.9770	9.5245	10.4755	30
40	9.5052	9.9765	9.5287	10.4713	20
50	9.5090	9.9761	9.5329	10.4671	10
19° 00′	9.5126	9.9757	9.5370	10.4630	71° 00′
10	9.5163	9.9752	9.5411	10.4589	50
20	9.5199	9.9748	9.5451	10.4549	40
30	9.5235	9.9743	9.5491	10.4509	30
40	9.5270	9.9739	9.5531	10.4469	20
50	9.5306	9.9734	9.5571	10.4429	10
20° 00′	9.5341	9.9730	9.5611	10.4389	70° 00′
10	9.5375	9.9725	9.5650	10.4350	50
20	9.5409	9.9721	9.5689	10.4311	40
30	9.5443	9.9716	9.5727	10.4273	30
40	9.54.7	9.9711	9.5766	10.4234	20
50	9.5510	9.9706	9.5804	10.4196	10
21° 00′	9.5543	9.9702	9.5842	10.4158	69° 00′
10	9.5576	9.9697	9.5879	10.4121	50
20	9.5609	9.9692	9.5917	10.4083	40
30	9.5641	9.9687	9.5954	10.4046	30
40	9.5673	9.9682	9.5991	10.4009	20
50	9.5704	9.9677	9.6028	10.3972	10
22° 00′	9.5736	9.9672	9.6064	10.3936	68° 00′
10	9.5767	9.9667	9.6100	10.3900	50
20	9.5798	9.9661	9.6136	10.3864	40
30	9.5828	9.9656	9.6172	10.3828	30
40	9.5859	9.9651	9.6208	10.3792	20
50	9.5889	9.9646	9.6243	10.3757	10
23° 00′	9.5919	9.9640	9.6279	10.3721	67° 00′
10	9.5948	9.9635	9.6314	10.3686	50
20	9.5978	9.9629	9.6348	10.3652	40
30	9.6007	9.9624	9.6383	10.3617	30
40	9.6036	9.9618	9.6417	10.3583	20
50	9.6065	9.9613	9.6452	10.3548	10
24° 00′	9.6093	9.9607	9.6486	10.3511	66° 00′
	L Cos	L Sin	L Cot	L Tan	Angle

* These tables give the logarithms increased by 10. Hence in each case 10 should be subtracted.

Table C: Logarithms of Trigonometric Functions*

Angle	L Sin	L Cos	L Tan	L Cot	
24° 00′	9.6093	9.9607	9.6486	10.3514	66° 00′
10	9.6121	9.9602	9.6520	10.3480	50
20	9.6149	9.9596	9.6553	10.3447	40
30	9.6177	9.9590	9.6587	10.3413	30
40	9.6205	9.9584	9.6620	10.3380	20
50	9.6232	9.9579	9.6654	10.3346	10
25° 00′	9.6259	9.9573	9.6687	10.3313	65° 00′
10	9.6286	9.9567	9.6720	10.3280	50
20	9.6313	9.9561	9.6752	10.3248	40
30	9.6340	9.9555	9.6785	10.3215	30
40	9.6366	9.9549	9.6817	10.3183	20
50	9.6392	9.9543	9.6850	10.3150	10
26° 00′	9.6418	9.9537	9.6882	10.3118	64° 00′
10	9.6444	9.9530	9.6914	10.3086	50
20	9.6470	9.9524	9.6946	10.3054	40
30	9.6495	9.9518	9.6977	10.3023	30
40	9.6521	9.9512	9.7009	10.2991	20
50	9.6546	9.9505	9.7040	10.2960	10
27° 00′	9.6570	9.9499	9.7072	10.2928	63° 00′
10	9.6595	9.9492	9.7103	10.2897	50
20	9.6620	9.9486	9.7134	10.2866	40
30	9.6644	9.9479	9.7165	10.2835	30
40	9.6668	9.9473	9.7196	10.2804	20
50	9.6692	9.9466	9.7226	10.2774	10
28° 00′	9.6716	9.9459	9.7257	10.2743	62° 00′
10	9.6740	9.9453	9.7287	10.2713	50
20	9.6763	9.9446	9.7317	10.2683	40
30	9.6787	9.9439	9.7348	10.2652	30
40	9.6810	9.9432	9.7378	10.2622	20
50	9.6833	9.9425	9.7408	10.2592	10
29° 00′	9.6856	9.9418	9.7438	10.2562	61° 00′
10	9.6878	9.9411	9.7467	10.2533	50
20	9.6901	9.9404	9.7497	10.2503	40
30	9.6923	9.9397	9.7526	10.2474	30
40	9.6946	9.9390	9.7556	10.2444	20
50	9.6968	9.9383	9.7585	10.2415	10
30° 00′	9.6990	9.9375	9.7614	10.2386	60° 00′
10	9.7012	9.9368	9.7644	10.2356	50
20	9.7033	9.9361	9.7673	10.2327	40
30	9.7055	9.9353	9.7701	10.2299	30
40	9.7076	9.9346	9.7730	10.2270	20
50	9.7097	9.9338	9.7759	10.2241	10
31° 00′	9.7118	9.9331	9.7788	10.2212	59° 00′
10	9.7139	9.9323	9.7816	10.2184	50
20	9.7160	9.9315	9.7845	10.2155	40
30	9.7181	9.9308	9.7873	10.2127	30
40	9.7201	9.9300	9.7902	10.2098	20
50	9.7222	9.9292	9.7930	10.2070	10
32° 00′	9.7242	9.9284	9.7958	10.2042	58° 00′
10	9.7262	9.9276	9.7986	10.2014	50
20	9.7282	9.9268	9.8014	10.1986	40
30	9.7302	9.9260	9.8042	10.1958	30
40	9.7322	9.9252	9.8070	10.1930	20
50	9.7342	9.9244	9.8097	10.1903	10
33° 00′	9.7361	9.9236	9.8125	10.1875	57° 00′
10	9.7380	9.9228	9.8153	10.1847	50
20	9.7400	9.9219	9.8180	10.1820	40
30	9.7419	9.9211	9.8208	10.1792	30
40	9.7438	9.9203	9.8235	10.1765	20
50	9.7457	9.9194	9.8263	10.1737	10
34° 00′	9.7476	9.9186	9.8290	10.1710	56° 00′
10	9.7494	9.9177	9.8317	10.1683	50
20	9.7513	9.9169	9.8344	10.1656	40
30	9.7531	9.9160	9.8371	10.1629	30
40	9.7550	9.9151	9.8398	10.1602	20
50	9.7568	9.9142	9.8425	10.1575	10
35° 00′	9.7586	9.9134	9.8452	10.1548	55° 00′
10	9.7604	9.9125	9.8479	10.1521	50
20	9.7622	9.9116	9.8506	10.1494	40
30	9.7640	9.9107	9.8533	10.1467	30
40	9.7657	9.9098	9.8559	10.1441	20
50	9.7675	9.9089	9.8586	10.1414	10
36° 00′	9.7692	9.9080	9.8613	10.1387	54° 00′
	L Cos	L Sin	L Cot	L Tan	Angle

Angle	L Sin	L Cos	L Tan	L Cot	
36° 00′	9.7692	9.9080	9.8613	10.1387	54° 00′
10	9.7710	9.9070	9.8639	10.1361	50
20	9.7727	9.9061	9.8666	10.1334	40
30	9.7744	9.9052	9.8692	10.1308	30
40	9.7761	9.9042	9.8718	10.1282	20
50	9.7778	9.9033	9.8745	10.1255	10
37° 00′	9.7795	9.9023	9.8771	10.1229	53° 00′
10	9.7811	9.9014	9.8797	10.1203	50
20	9.7828	9.9004	9.8824	10.1176	40
30	9.7844	9.8995	9.8850	10.1150	30
40	9.7861	9.8985	9.8876	10.1124	20
50	9.7877	9.8975	9.8902	10.1098	10
38° 00′	9.7893	9.8965	9.8928	10.1072	52° 00′
10	9.7910	9.8955	9.8954	10.1046	50
20	9.7926	9.8945	9.8980	10.1020	40
30	9.7941	9.8935	9.9006	10.0994	30
40	9.7957	9.8925	9.9032	10.0968	20
50	9.7973	9.8915	9.9058	10.0942	10
39° 00′	9.7989	9.8905	9.9084	10.0916	51° 00′
10	9.8004	9.8895	9.9110	10.0890	50
20	9.8020	9.8884	9.9135	10.0865	40
30	9.8035	9.8874	9.9161	10.0839	30
40	9.8050	9.8864	9.9187	10.0813	20
50	9.8066	9.8853	9.9212	10.0788	10
40° 00′	9.8081	9.8843	9.9238	10.0762	50° 00′
10	9.8096	9.8832	9.9264	10.0736	50
20	9.8111	9.8821	9.9289	10.0711	40
30	9.8125	9.8810	9.9315	10.0685	30
40	9.8140	9.8800	9.9341	10.0659	20
50	9.8155	9.8789	9.9366	10.0634	10
41° 00′	9.8169	9.8778	9.9392	10.0608	49° 00′
10	9.8184	9.8767	9.9417	10.0583	50
20	9.8198	9.8756	9.9443	10.0557	40
30	9.8213	9.8745	9.9468	10.0532	30
40	9.8227	9.8733	9.9494	10.0506	20
50	9.8241	9.8722	9.9519	10.0481	10
42° 00′	9.8255	9.8711	9.9544	10.0456	48° 00′
10	9.8269	9.8699	9.9570	10.0430	50
20	9.8283	9.8688	9.9595	10.0405	40
30	9.8297	9.8676	9.9621	10.0379	30
40	9.8311	9.8665	9.9646	10.0354	20
50	9.8324	9.8653	9.9671	10.0329	10
43° 00′	9.8338	9.8641	9.9697	10.0303	47° 00′
10	9.8351	9.8629	9.9722	10.0278	50
20	9.8365	9.8618	9.9747	10.0253	40
30	9.8378	9.8606	9.9772	10.0228	30
40	9.8391	9.8594	9.9798	10.0202	20
50	9.8405	9.8582	9.9823	10.0177	10
44° 00′	9.8418	9.8569	9.9848	10.0152	46° 00′
10	9.8431	9.8557	9.9874	10.0126	50
20	9.8444	9.8545	9.9899	10.0101	40
30	9.8457	9.8532	9.9924	10.0076	30
40	9.8469	9.8520	9.9949	10.0051	20
50	9.8482	9.8507	9.9975	10.0025	10
45° 00′	9.8495	9.8495	10.0000	10.0000	45° 00′
	L Cos	L Sin	L Cot	L Tan	Angle

* These tables give the logarithms increased by 10. Hence in each case 10 should be subtracted

Answers to the following questions are to be written on paper provided by the school.

Part II

Answer four questions from this part. Show all work unless otherwise directed. [40]

36 *a* Express the roots of the equation $2x^2 = -3(2x + 3)$ in $a + bi$ form. [5]

b Using logarithms, solve the equation $2^{3x} = 7$ for x to the *nearest tenth*. [5]

37 *a* Sketch and label the graph of the equation $y = 3^x$. [4]

b The graph of the equation $y = 3^x$ is reflected in the y-axis. On the same set of axes used in part *a*, sketch this reflection. [4]

c Which is an equation of the reflection sketched in part *b*? [2]

(1) $y = -(3^x)$ (3) $y = \log_3 x$

(2) $y = 3^{-x}$ (4) $x = 3^y$

38 *a* On the same set of axes, sketch and label the graphs of the equations $y = 2 \sin \frac{1}{2}x$ and $y = \cos 2x$ in the interval $0 \leq x \leq 2\pi$. [8]

b What is a value of x in the interval $0 \leq x \leq 2\pi$ for which $2 \sin \frac{1}{2}x - \cos 2x = 1$? [2]

(1) 0 (3) π

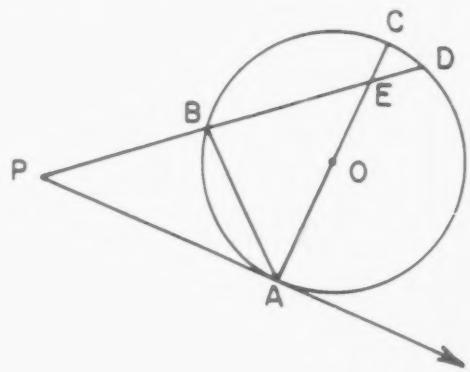
(2) $\frac{\pi}{2}$ (4) $\frac{3\pi}{2}$

39 *a* Find all values of θ in the interval $0° \leq \theta < 360°$ which satisfy the equation $2 \sin \theta - 1 = \csc \theta$. [6]

b For all values of θ for which the expressions are defined, prove the following is an identity:

$$\frac{\cos \theta \sin \theta + \cos \theta}{\cos^2 \theta} = \tan \theta + \sec \theta$$ [4]

40 In the accompanying diagram, $\overrightarrow{PA}$ is tangent to circle O at point A, secant $\overline{PBD}$ intersects diameter $\overline{AC}$ at point E, $m\angle P = 40$, and $m\overset{\frown}{CD} : m\overset{\frown}{DA} = 1:8$.



Find:

a $m\overset{\frown}{AD}$ [2]

b $m\overset{\frown}{AB}$ [2]

c $m\angle BEA$ [2]

d $m\angle BAC$ [2]

e $m\angle PBA$ [2]

GO RIGHT ON TO THE NEXT PAGE.

41 An electronic game contains 9 keys. Four of the keys have numerals and five have symbols. Each key is equally likely to be pressed on each move.

1	2	∧
3	4	∨
-	<	>

a Find:
(1) P (4) [1]
(2) P (numeral key) [1]
(3) P (symbol key) [1]

b Find the probability of:
(1) pressing *exactly* 2 numeral keys on 3 random presses [3]
(2) pressing *at least* 2 symbol keys on 3 random presses [4]

42 The table below represents the weight, in pounds, of the students in Mrs. Grabenstein's homeroom.

x_i	f_i
68	4
76	4
80	3
82	6
83	2
86	1

Using this set of data, find

a the mean [2]
b the median [1]
c the mode [1]
d the standard deviation to the *nearest tenth* [6]

The University of the State of New York

REGENTS HIGH SCHOOL EXAMINATION

SEQUENTIAL MATH — COURSE III

Monday, June 23, 1986 — 1:15 to 4:15 p.m., only

Part I Score	
Part II Score	
Total Score	
Rater's Initials	

ANSWER SHEET

Pupil ..... Teacher .....

School ..... Grade .....

Your answers to Part I should be recorded on this answer sheet.

Part I

Answer 30 questions from this part.

1	11	21	31 .....
2	12	22	32 .....
3	13	23	33 .....
4	14	24	34 .....
5	15	25	35 .....
6	16	26	
7	17	27	
8	18	28	
9	19	29	
10	20	30	

Your answers for Part II should be placed on paper provided by the school.

The declaration below should be signed when you have completed the examination.

I do hereby affirm, at the close of this examination, that I had no unlawful knowledge of the questions or answers prior to the examination, and that I have neither given nor received assistance in answering any of the questions during the examination.

Signature

SEQUENTIAL MATHEMATICS COURSE III
REGENTS QUESTIONS WITH A TWIST

1. If P (-2, 4) is reflected over the line $y = -x$, what are the coordinates of P', the image of P?

2. Three fair coins are tossed. What is the probability of obtaining <u>at least</u> two tails?

3. Given the following table of SAT scores, whta is the range of these scores?

Score	210	370	640	700	790
Frequency	1	5	7	4	1

4. Using logarithms, find $\sqrt[3]{0.692}$ to the <u>nearest hundredth.</u>

5. Find the largest integral value of k in the equation $kx^2 - 5x + 3 = 0$ that will make its roots real.

6. Using logarithms, solve the equation $4^x = 70$ for x, correct to the <u>nearest tenth</u> .

7. Find the third term of the expansion of $(x-2)^6$. Write your answer in simplest form.

8. Express in simplest form: $\dfrac{\frac{1}{x^2} - 1}{1 + \frac{1}{x}}$

9. Combine and reduce to **lowest terms:**

$$\frac{4x}{2x+6} + \frac{x}{x+3}$$

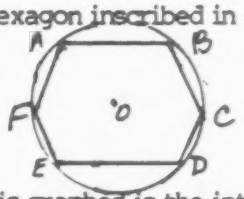
10. If $\log N = \log A + 3\log b$, solve for N in terms of A and B.

11. If $1 + i$ and $1 - i$ are the roots of the quadratic equation $x^2 + bx + c = 0$, what is the value of c?

12. Express in **simplest form:** $\frac{9}{\sqrt{27}} + \sqrt{12}$

13. ABCDEF is a regular hexagon inscribed in circle O. With O as the center of rotation, find:

$\text{Rot}_{(-120^\circ)} \circ R_{240^\circ}$ (A)

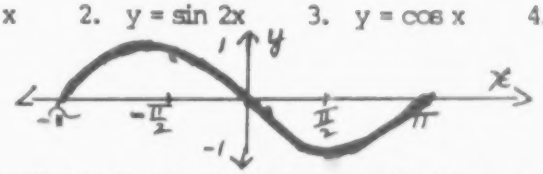


14. The equation $y = \tan x$ is graphed is the interval $0 \leq x \leq \frac{\pi}{2}$ and is reflected over the x-axis. On this reflection, point $(\frac{\pi}{4}, y)$ has what value for y?

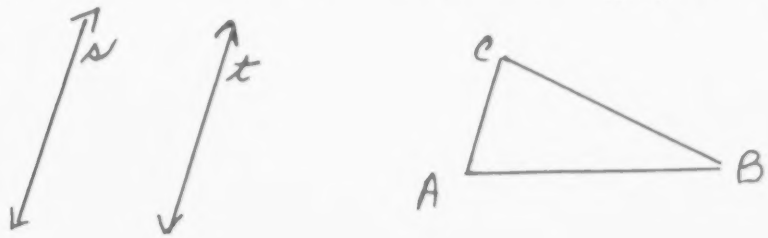
15. Under composition, the identity function $y = x$ is its own inverse. Name another function is the form $y = mx + b$ that is its own inverse.

16. The graph is the accompanying diagram is a reflection in the y-axis of which equation?

1. $y = \sin x$ 2. $y = \sin 2x$ 3. $y = \cos x$ 4. $y = \cos 2x$

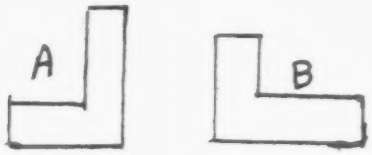


17. Line s is parallel to line t. Which is equivalent to $r_s \circ r_t$ (ABC)?



18. Figure B is the image of figure A under which single transformation?

1. line reflection 2. translation 3. rotation 4. glide reflection



19. Find the solution set: $x = \sqrt{3x + 4}$

20. Evaluate $\sum_{K=1}^{3} \frac{1}{2}(k^2)$.

21. Find the value of 2(Arc sin 1).

22. Find the value of i^{16}.

23. If f (-2) = 8, find f^{-1} (8)

24. Under a translation, the image of (3,-6) is (1, -3). Fine the pre-image of (5,2) under the same translation.

25. Which one of the suits used in a full deck of playing cards has two lines of symmetry?

26. Using h (x) = x^2 and g (x) = 2 x -3, find h o g (-3)

27. If $\log_3 n - \log_3 \frac{1}{3} = \log_3 9$, what is the value of n?

28 If log 3 = a, find log 90 in terms of a

29. Use the "Σ" symbol to write an expression for $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

30. Find $\sin\left(\text{Arc tan}\left(-\frac{3}{4}\right)\right)$

REFERENCES AND RESOURCES FOR TEACHERS OF COURSE III

1. BUMBY, D. AND KLUTCH, R.: MATHEMATICS A TOPICAL APPROACH -COURSE 3 MERRILL, 1980.

2. CREATIVE PUBLICATIONS POSTERS - MATHEMATICS IN NATURE, BUTTERFLY POSTERS, STAINED GLASS TESSELATIONS.

3. JACOBS, H.: MATHEMATICS, A HUMAN ENDEAVOR 2ND EDITION FREEMAN & CO.,1982.

4. KEENAN, E. AND GANTERT, A: INTEGRATED MATHEMATICS - COURSE III AMSCO, 1982.

5. KEENAN E. et al: DEVELOPMENT AND DEMONSTRATION HANDBOOK SEWANANKA CENTRAL HIGH SCHOOL DISTRICT, HERRICKS PUBLIC SCHOOLS, NASSAU COUNTY, NEW YORK. TITLE IV-C DEMONSTRATION GRANT BOOKLET, 1979.

6. NEW YORK STATE EDUCATION DEPARTMENT CURRICULUM PUBLICATION: THREE YEAR SEQUENCE FOR HIGH SCHOOL MATHEMATICS - COURSE III, 1979

*7. OTT, J. et al, TEACHER RESOURCE BOOK - MATHEMATICS A TOPICAL APPROACH COURSE 3 MERRILL, 1986.

8. POSAMENTIER, A. AND STEPELMAN, J.: TEACHING SECONDARY SCHOOL MATHEMATICS MERRILL, 1980.

9. RISING, G. et al: UNIFIED MATHEMATICS - BOOK 3 HOUGHTON MIFFLIN, 1982.

10. USISKIN, Z.: ONTARIO MATHEMATICS GAZETTE; THE CASE FOR TRANSFORMATIONS IN HIGH SCHOOL GEOMETRY, P. 40-44. DEC., 1978.

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